Charles University in Prague
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MASTER THESIS

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Konstrukce minimálních DNF reprezentací 2-intervalových funkcí

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and Mathematical Logic

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I devote this master thesis to my parents for giving me an opportunity to study which I greatly appreciate. I would like to say real thanks to my supervisor doc. Ondřej Čepek. His guidance through all stages of work from detailed understanding of a problem to scatching an outline was of great value to me. I am also thankful to my friend Jan Rosík for creating all pictures.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In .... date .... Jakub Dubovský

Title: A construction of minimum DNF representations of 2-interval functions

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Abstract: The thesis is devoted to interval boolean functions. It is focused on construction of their representation by disjunctive normal forms with minimum number of terms. Summary of known results in this field for 1-interval functions is presented. It shows that method used to prove those results cannot be in general used for two or more interval functions. It tries to extend those results to 2-interval functions. An optimization algorithm for special subclass of them is constructed. Exact error estimation for approximation algorithm is proven. A command line software for experimentation with interval function is part of the thesis.

Keywords: boolean function, interval function, representation construction, approximation

Název práce: Konstrukce minimálních DNF reprezentací 2-intervalových funkcí

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Abstrakt: Tato práce se věnuje intervalovým boolovským funkcím. Je zaměřena na konstrukci jejich reprezentací pomocí disjunktivních normálních forem s co nejmenším počtem termů. Shrnuje známé výsledky v této oblasti pro 1-intervalové funkce. Ukazuje, že používanou metodu důkazu nelze obecně použít pro dva a více intervalové funkce. Pokouší se rozšířit tyto poznatky na 2-intervalové funkce. Je navrhnut optimalizační algoritmus pro speciální podtřídu 2-intervalových funkcí. Dokazuje se přesný odhad chyby pro jednoduchý aproximační algoritmus. Součástí práce je softwerová utilita pro experimentování s intervalovými funkcemi.

Klíčová slova: boolovská funkce, intervalová funkce, konstrukce reprezentace, aproximace

Contents

1. Introduction

Boolean function is an important notion within numerous fields of theoretical informatics. Even though it is widely studied there are still many interesting open problems waiting to be adressed. In this thesis we will focus on a special subclass of boolean functions called interval functions and their representations. For a more detailed description of the problem studied in this thesis we need to present some basic definitions first.

1.1 Definitions

We begin with some essential definitions.

Definition 1.1 (Boolean function)**.** *A boolean function (or function for short) on n propositional variables is a mapping $f : \{0,1\}^n \to \{0,1\}$. The number of variables can be explicitly stated by writing f^n.*

We will omit the adjective "boolean" where it is clear what kind of a function we refer to. Values 0 and 1 are usualy refered to as $false$ and $true$. We can concatenate them together to get a boolean vector.

Definition 1.2 (Boolean vector)**.** *A boolean vector x (or a vector for short) is an n-tuple of boolean values $\{0,1\}$.*

Definition 1.3 (Complement of a boolean vector)**.** *The complement of a boolean vector x is a boolean vector denoted by $\bar{x}$ of the same length with the property that they differ on every position.*

We are going to define relational operators $(<,>,\leq,\geq,=)$ on boolean vectors. It is worth of a little note here. In this thesis we do not distinquish between a boolean vector and a number whose binary representation the vector is. A convention is that the most significant bit (MSB) is the first bit of the vector and the least significant bit (LSB) is the last bit of the vector. We will work with boolean vectors but we will compare them as numbers represented by them. This will become natural after we reach the definition of interval function.

Definition 1.4 (Relational operators)**.** *Let $R \in \{<,>,\leq,\geq,=\}$ be a relational operator. Then define relation xRy between two boolean vectors to be the same as xRy when applied on numbers.*

Definition 1.5 (Truepoint)**.** *A boolean vector (a number) x is called a truepoint of function f if $f(x) = 1$.*

Definition 1.6 (Falsepoint)**.** *A boolean vector (a number) x is called a falsepoint of function f if $f(x) = 0$.*

Definition 1.7 (Partial assignment)**.** *Let f^n be a boolean function, x_i one of its variables and c be a boolean constant. We denote by $f[x_i := c]$ the function of $n-1$ variables with the following property:*

$$f(x_1 \dots x_{i-1}, c, x_{i+1} \dots x_n) = f[x_i := c](x_1 \dots x_{i-1}, x_{i+1} \dots x_n)$$

1.2 Representations of boolean functions

Probably the most basic representation is a result of the observation that the domain of a boolean function is finite. That enables us to simply enumerate all function values.

Definition 1.8 (Truth table)**.** *The truth table for function f^n is the table with 2^n rows each of them containing a unique boolean vector x and function value $f(x)$.*

A common way of representing a function is by logical operators. Those are simple function usually on one or two variables defined by truth tables. Their combinations are called logical formulas. The most important operators are $(\neg, \wedge, \vee)$ named logical negation, logical conjuction and logical disjunction respectively. They form a basis which means that for every boolean function there is at least one formula composed from those operators which represents the function.

x	y	$\neg x$	$x \wedge y$	$x \vee y$
1	1	0	1	1
0	1	1	0	1
1	0	0	0	1
0	0	1	0	0

Figure 1.1: Truthtable for negation, conjunction and disjunction

There are widely used subclasses of logical formulas called *normal forms*. These subclasses can still represent every function but they are smaller because of syntactic restrictions they impose on the allowed form of formulas.

Definition 1.9 (Literal)**.** *A literal is a boolean variable x and its negation $\neg x$ (positive and negative literal resp.). For convenience we will write $\bar{x}$ instead of $\neg x$.*

Definition 1.10 (Term)**.** *A term t is an conjuction of literals where every propositional variable appears in it at most once, i.e., if $I \cap J = \emptyset$*

$$t = \bigwedge_{i \in I} x_i \wedge \bigwedge_{j \in J} \bar{x}_j \tag{1.1}$$

Definition 1.11 (Disjunctive normal form)**.** *A disjunctive normal form (or DNF) is a disjunction of terms.*

Lemma 1.1. *Every boolean function f can be represented by a DNF.*

Proof. We will construct a term from every truepoint of f as follows. Let $\mathbf{x}$ be a truepoint. Put x into a term $t(x)$ for every variable which is true in $\mathbf{x}$. Put $\bar{x}$ into a term $t(x)$ for every variable which is false in $\mathbf{x}$. Such a term $t(x)$ is true only on truepoint x used to construct it. Disjunction $\mathscr{T}$ of such terms $t(x)$ for all truepoints x is a DNF representation of f. Each truepoint of f has its own term in $\mathscr{T}$ so it evaluates to 1. On the other hand each false point of f differs from all truepoints of f so there is no term which is evalutated to 1. Therefore $\mathscr{T}$ is evaluated to 0. □

Another possible representation of boolean function is a perfect binary tree of depth n. In such a tree each leaf is in depth n so all inner nodes have two children. Mark the left one with 0 and the right one with 1. Each leaf has assigned a boolean vector of length n as the sequence of zeros and ones on the path from the root to the leaf. No two leafs have the same vector. All vectors from $\{0,1\}^n$ are assigned to some leaf. Moreover assign the value $f(x)$ to the leaf representing vector x. There are more branching trees representations to one function. It depends on the assigment of variables to levels of the tree. We will use this representation informally to illustrate the ideas presented in this thesis.

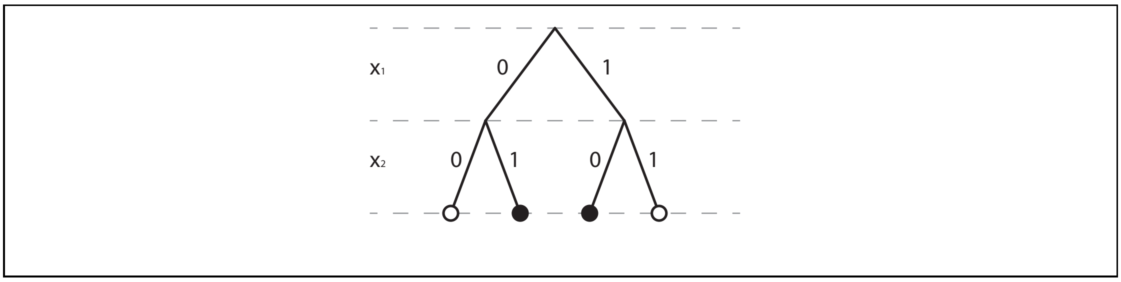

Figure 1.2: Branching tree for function "Exclusive OR"

We have described representations we will use in this thesis. However, there are many more possible representations. For more detailed information see [1]

1.3 Functions defined by intervals

Now we can define interval functions. In fact the notion is parameterized by an integer. There exist i-interval functions for $i \in \{0,1,2,\dots\}$. However we do not need the general definition. It is enough to define the two cases we will actually use.

Definition 1.12 (1-interval function). *Let $a \leq b$ be two n-bit binary numbers. Then $f^n_{[a,b]} : \{0,1\}^n \to \{0,1\}$ is an 1-interval function defined as follows:*

$$f^n_{[a,b]}(x) = \begin{cases} 1 & \textit{if } x \textit{ is the number in the interval [a,b]} \\ 0 & \textit{otherwise} \end{cases}$$

Definition 1.13 (2-interval function). *Let $a \leq b < c \leq d$ be four n-bit binary numbers. Then $f^n_{[a,b],[c,d]} : \{0,1\}^n \to \{0,1\}$ is an 2-interval function defined as follows:*

$$f^n_{[a,b],[c,d]}(x) = \begin{cases} 1 & \textit{if } x \textit{ is the number from an interval [a,b] or [c,d]} \\ 0 & \textit{otherwise} \end{cases}$$

For an example of 1-interval function we can look at basic functions. Disjunction of two variables can be seen as function $f^2_{[1,3]}$. Exlusive disjunction in figure 1.2 is $f^2_{[1,2]}$. Example of 2-interval function can be seen in figure 1.3. Note that every 1-interval function with at least two truepoints is 2-interval as well ($f^2_{[1,2]}$ is same as $f^2_{[1,1],[2,2]}$).

Interval function f^n defined by k intervals can be represented by $2k+1$ n-bit numbers. Two for every interval and one for the number of variables (note the

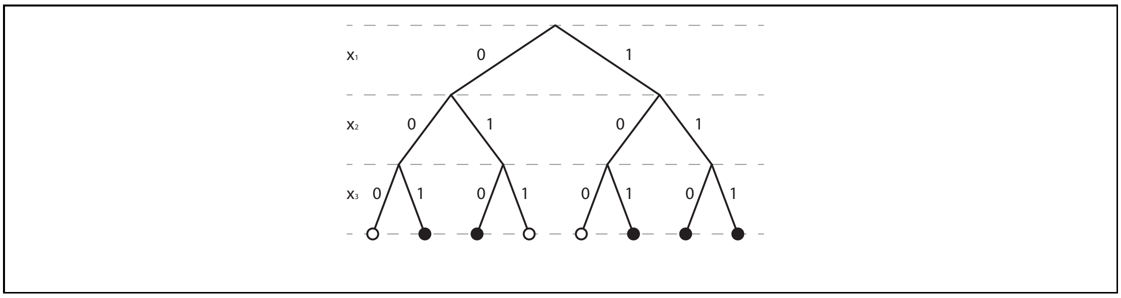

Figure 1.3: Example of two interval function: $f^3_{[1,2],[5,7]}$

difference between $f^3_{[1,4]}$ and $f^4_{[1,4]}$). However in a real world the above mentioned representation often needs to be converted to some uniform representation such as a DNF. Moreover the resulting DNF needs to be of a minimum size measured by the number of terms. The problem of finding a minimum DNF for a given 2-interval function is the problem adressed by this thesis.

We will partition the class of 2-interval functions into three disjunct subclasses. The division is done by function values of two vectors. First consisting only from 0-bits and second only from 1-bits.

- A_0 is the class of 2-interval functions where $a = 0$ and $d = 2^n - 1$.
- B_0 is the class of 2-itnerval functions where exactly one of the following holds: $a = 0$ or $d = 2^n - 1$.
- C_0 is the class of 2-interval functions where $a \neq 0$ and $d \neq 2^n - 1$.

Now we define an alternative way of looking at the problem. Let $T = t^{[1]} \dots t^{[n]}$ be a *ternary vector* of length n over an alphabet $\{0, 1, \phi\}$. When there is no chance of confusion between boolean and ternary vector we say just a vector. We say that T *spans* all binary vectors $B = b^{[1]} \dots b^{[n]}$ for which $b^{[i]} = t^{[i]}$ for all i such that $t^{[i]} \in \{0, 1\}$. So the ternary vector is like a mask where we treat the symbol ϕ as "don't care". We denote the set of numbers spanned by vector T as $dom(T)$. Let $\mathscr{T}$ be the set of ternary vectors of length n. Binary vectors spanned by $\mathscr{T}$ are defined as a union of vectors spaned by each of the vectors in $\mathscr{T}$ and it is denoted by $dom(\mathscr{T})$.

Definition 1.14 (Spanning set)**.** *Set of ternary vectors spanning exactly the truepoints of function f is called a spanning set of f.*

Let f^n be the function represented by DNF $\mathscr{F}$. Then we can produce a spanning set of f directly from $\mathscr{F}$. Fix the order of variables. Then each term gives a ternary vector of length n. We construct a ternary vector from term t as follows:

$$T^{[i]} = \begin{cases} 1 & \text{if } x_i \text{ appears as positive literal in term t} \\ 0 & \text{if } x_i \text{ appears as negative literal in term t} \\ \phi & \text{if } x_i \text{ does not appear in term t} \end{cases}$$

It is clear that such a T spans exactely those numbers which satisfies term t. Boolean vector is a truepoint of f if it satisfies some term in the same way as it is a truepoint if it is spanned by some ternary vector. The union of ternary vectors constructed form terms of $\mathscr{F}$ is a spanning set of f^n.

Reversing the process we get the procedure for conversion of a spanning set into a DNF without changing the function being represented. The number of terms is equal to number of vectors after conversion in both directions. And so we see that problem of finding a minimum DNF representation of a function is equivalent to the problem of finding a minimum spanning set of a function. We will consider this problem in the rest of the thesis. We will say sometimes that we are spanning an interval $[a, b]$ instead of a function $f^n_{[a,b]}$.

Definition 1.15 (Orthogonal set)**.** *Let f^n be a boolean function of n variables. Let e_1 and e_2 be truepoints of f^n. We call them orthogonal given f^n (for f^n) if every ternary vector of length n spanning both e_1 and e_2 necessarily spans also some falsepoint of f^n.*

We say that a k-tuple of truepoints of f^n is an orthogonal set given f^n (for f^n) if every pair of truepoints from it is orthogonal given f^n (for f^n). We will omit "given f^n" or "for f^n" when it is clear which function we refer to.

Immediately from this definition we have the observations which gives us a way how to prove the minimality of spanning set.

Observation 1.1. *The size of an orthogonal set for function f^n is a lower bound on the size of a minimum spanning set of f^n.*

Observation 1.2. *Let $\mathscr{T}$ be a spanning set of f^n. If $\mathscr{T}$ has a size of some orthogonal set for f^n then $\mathscr{T}$ is an minimum spanning set of f^n.*

After these observations it is natural to raise a question if following hypothesis holds.

Hypothesis 1.1. *The size of a minimum spanning set of f^n is the same as the size of a maximum orthogonal set for f^n.*

Note that we formulated this hypothesis for general boolean functions not just for i-interval functions for some i. We will try to answer this question in the following sections.

2. Spanning sets of 1-interval functions

Constructing the minimum spanning set of 1-interval function is already described and understood (see[2]). This thesis tries to study this problem on the class of 2-interval functions. In this chapter we summarize the results presented in [2] which are used further in the text. While ideas are preserved we took slightly different approach to their explanation. That is thanks to different notations. We need to start by some notation which will be used.

To refer to a particular bit of a vector we use brackets in superscript like this $a = a^{[1]}a^{[2]}\dots a^{[n]}$. First bit has an index one. Extraction of range of bits we denote by $a^{[2,4]} = a^{[2]}a^{[3]}a^{[4]}$. When we want to iterate particular bit we write $\phi^{\{3\}} = \phi\phi\phi$. All of these constructions can be concatenated to create a new vector. For example when $a = 001\phi$ then $a^{[2,4]}0\phi^{\{2\}}$ means the vector $01\phi0\phi\phi$. Let T be a ternary vector. Denote by $comp(T)$ its complement. Define it by 1-complementing 0-bits and 1-bits and leaving ϕ-bits unchanged.

2.1 Prefix and suffix cases

Prefix $f^n_{[0,b]}$ and suffix $f^n_{[a,2^n-1]}$ cases are solved first.

Lemma 2.1. *(Prefix case) The minimum number of ternary vectors needed to span a prefix function $f^n_{[0,b]}$ is the same as the number of 1-bits in number $(b+1)$.*

Proof. If $b = 2^n - 1$ than we can span the function $f^n_{[0,2^n-1]}$ by single vector $\phi^{\{n\}}$ which is indeed a minimum spaning set. Assuming that $b < 2^n - 1$ let $c = b + 1$. Denote by k the number of 1-bits in c and by $o_1, \dots, o_k$ indeses of those 1-bits. Construct the set of binary vectors $V = \{V_j | 1 \le j \le k\}$ where:

$$V_j = c^{[1,o_j-1]}0c^{[o_j+1,n]} \tag{2.1}$$

Let T be a ternary vector spanning V_i and V_j $(i \ne j)$. Since $V_j^{[o_j]} = 0$ and $V_i^{[o_j]} = 1$ it has to be that $T^{[o_j]} = \phi$. The same holds for $T^{[o_i]} = \phi$. On the rest of indeses both V_i and V_j are equal to number c. All this implies that vector T spans falsepoint c as well. We have proven that V is an orthogonal set for function $f^n_{[0,b]}$.

Now construct the spanning set of $f^n_{[0,b]}$. Denote it by $\mathscr{T} = \{T_j | 1 \le j \le k\}$ where:

$$T_j = c^{[1,o_j-1]}0\phi^{\{n-o_j\}} \tag{2.2}$$

It is easy to see that every number spanned by $\mathscr{T}$ is less than c and thus belongs to $[0, b]$. On the other hand consider number d from $[0, b]$. Since $d \ne c$ there are some positions where d and c differs. Because of $d < c$ it has to be the case that for the least such an index i holds $d^{[i]} = 0$ and $c^{[i]} = 1$. And so number d is spanned by vector T_j where $o_j = i$.

Clearly $|\mathscr{T}| = k = |V|$. We have constructed a spanning set of $f^n_{[0,b]}$ and an orthogonal set for $f^n_{[0,b]}$ of the same size. By the observation 1.2 $\mathscr{T}$ is a minimum spanning set. ◻

The interval $[0, b]$ was divided into smaller subintervals each spanned by one ternary vector. This division is in the direct relationship with the way of decomposition the number of truepoints in the interval (its length) into sum of powers of two. That is why each ternary vector of constructed spanning set corresponds to 1-bit in the number $b+1 = c$ which is a number of truepoints in prefix interval $[0, b]$. An orthogonal set constructed by this proof will be referenced by numerous proofs in remaining text.

Lemma 2.2. *(Suffix case) For $0 < a$ the minimum number of ternary vectors needed to span the suffix function $f^n_{[a,2^n-1]}$ is the same as the number of 0-bits in number $(a-1)$.*

Proof. It is enough to note that complementing the vectors in spanning set of $f^n_{[0,b]}$ we get the spanning set of the function $f^n_{[\bar{b},2^n-1]}$. Thus we can construct the spanning set of suffix function by complementing it, using lemma 2.1 and then complement it back. $\square$

2.2 The general case

The general algorithm generating spanning set for 1-interval function is decribed in this section. The minimality is proved then in the standalone theorem. We start by a definition.

Definition 2.1 (Cyclic shifts of a vector)**.** *Let T be a ternary vector of length n. Construct n vectors as follows:*

$$T_i = \begin{cases} T & i = 1 \\ T^{[n]}T^{[1]}\ldots T^{[n-1]} & i = 2 \\ T^{[n-i+2]}\ldots T^{[n]}T^{[1]}\ldots T^{[n-i+1]} & 2 < i < n \\ T^{[2]}\ldots T^{[n]}T^{[1]} & i = n \end{cases}$$

We say that vectors $T_1, \ldots, T_n$ are cyclic shifts of a vector T.

Algorithm is recursive in number of bits n and it uses lemma 2.1 and lemma 2.2 to deal with prefix and suffix cases. Also the solution is trivial when $n \leq 2$. Thus it is assumed that $0 < a$, $b < 2^n - 1$ and $n \geq 3$. The case $a^{[1]} = 1 \wedge b^{[1]} = 0$ is not possible since $a \leq b$. We can solve the case $a^{[1]} = b^{[1]}$ by computing spanning set of function definted by interval $[a^{[2,n]}, b^{[2,n]}]$ and then add a leading bit $a^{[1]}$ to each vector. So it is assumed without loss of generality that $a^{[1]} = 0$ and $b^{[1]} = 1$. We distinguish four cases now (two of them being symmetric) depending on first two MSBs of numbers a and b.

Case 1: We have $a^{[1,2]} = 01 \wedge b^{[1,2]} = 10$. Let $a' = a^{[3,n]}$ and $b' = b^{[3,n]}$. Interval $[a, b]$ can be partitioned into two subintervals $[a, 2^{n-1} - 1]$ and $[2^{n-1}, 2^n - 1]$. Using lemma 2.2 we find a set of vectors $\mathscr{T}_1$ spanning the suffix function $f^{n-2}_{[a',2^{n-2}-1]}$. Similarly using lemma 2.1 we find a set of vectors $\mathscr{T}_2$ spanning the prefix function $f^{n-2}_{[0,b']}$. We get the set $\mathscr{T}$ that spans the function $f^n_{[a,b]}$ as

$$\{01T | T \in \mathscr{T}_1\} \cup \{10T | T \in \mathscr{T}_2\} \tag{2.3}$$

Indeed the constructed set is the spanning set of the function $f^n_{[a,b]}$.

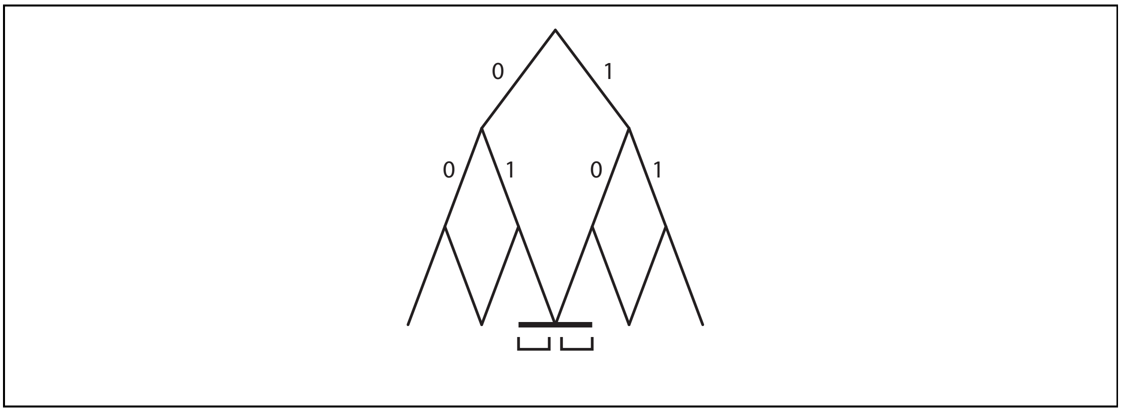

Figure 2.1: Case 1: Truepoint interval position and its partitioning into subintervals used in proof

Case 2: We have $a^{[1,2]} = 00 \wedge b^{[1,2]} = 10$. Let $a' = a^{[1]}a^{[3,n]}$ and $b' = b^{[1]}b^{[3,n]}$. Interval $[a, b]$ can be partitioned into three subintervals $[a, 2^{n-2}-1]$, $[2^{n-2}, 2^{n-1}-1]$ and $[2^{n-1}, b]$. Compute recursively set of vectors $\mathscr{T}'$ spanning a function $f^{n-1}_{[a',b']}$. We construct the spanning set as follows:

$$\mathscr{T} = \{T^{[1]}0T^{[3,n]} | T \in \mathscr{T}'\} \cup \{01\phi^{\{n-2\}}\} \tag{2.4}$$

It is not hard to see that constructed set is a spanning set of a function $f^n_{[a,b]}$.

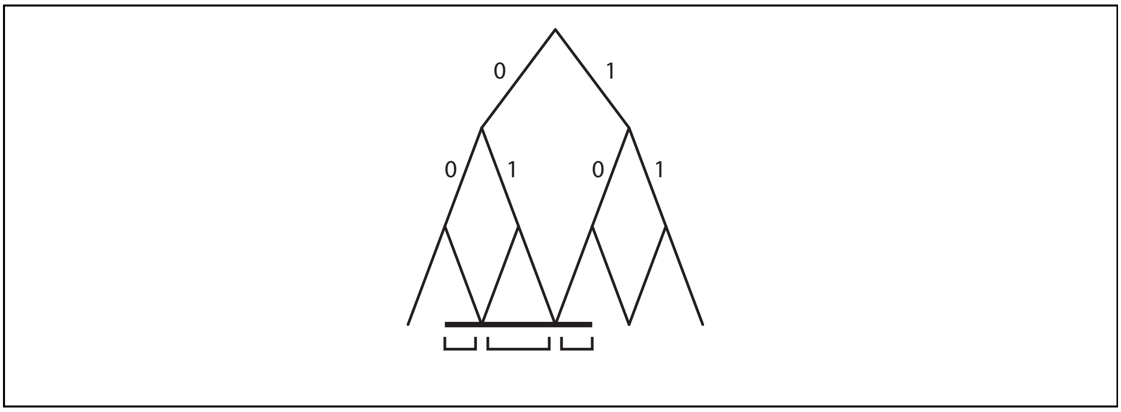

Figure 2.2: Case 2: Truepoint interval position and its partitioning into subintervals used in proof

Case 3: We have $a^{[1,2]} = 01 \wedge b^{[1,2]} = 11$. This is a complement of the case 2 and it is done similarly.

Case 4: We have $a^{[1,2]} = 00 \wedge b^{[1,2]} = 11$. Let j be the maximum integer such that $a^{[1,j]} = 0^{\{j\}}$ and $b^{[1,j]} = 1^{\{j\}}$. Let a' be the $(n-j)$-bit number $a^{[j+1,n]}$ and b' be the $(n-j)$-bit number $b^{[j+1,n]}$. Note that $a' < 2^{n-j}$ and $b' > 2^n - 2^{n-j} - 1$. The function $f^n_{[a,b]}$ can be partitioned into three subfunctions $f^n_{[a,a_1-1]}$, $f^n_{[a_1,b_1-1]}$ and $f^n_{[b_1,b]}$ where $a_1 = 2^{n-j}$ and $b_1 = 2^n - 2^{n-j}$. The idea now is to span the second subfunction using j ternary vectors and to span the other two subfunctions recursively. In a case when $b' \geq a' - 1$ the vectors spanning first and third subfunction overlap in such a way that it is possible to span the second subfunction using only $j - 1$ vectors. We will distinguish these cases later.

Now we show how to span the second subfunction $f^n_{[a_1,b_1-1]}$ by j vectors. Notice that its interval consists of exactly those numbers whose j MSBs contains at least

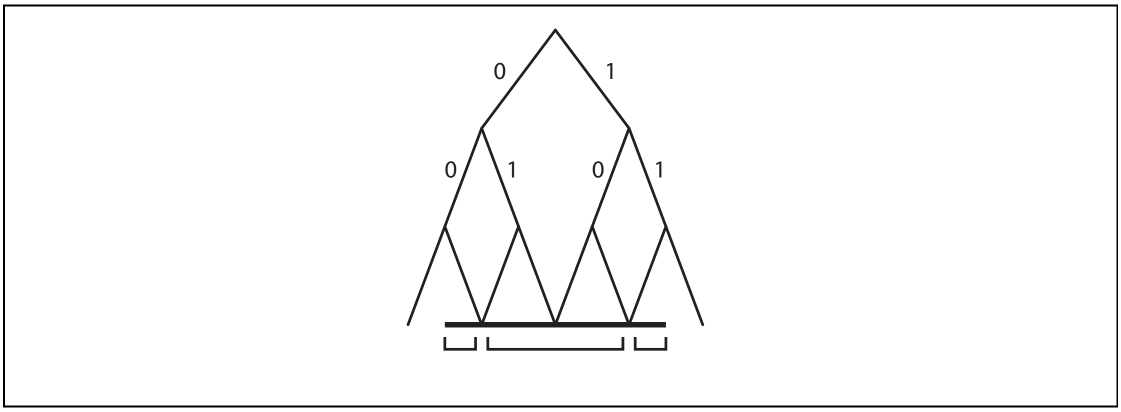

Figure 2.3: Case 4: Truepoint interval position and its partitioning into subintervals used in proof

one 0-bit and at least one 1-bit. Therefore for each number B in the interval there exists a position $i = i(B)(1 \leq i \leq j)$ such that $b^{[i]} = 0$ and $b^{[1+((i+1) \bmod j)]} = 1$. Let $T'_1, \dots, T'_j$ be all the cyclic shifts of the vector $01\phi^{\{j-2\}}$. Let $T_1, \dots, T_j$ be the j ternary vectors of length n given by concatenating each of the vectors T'_i with $n-j$ trailing ϕ-bits. We conslude that every number $B \in [a_1, b_1 - 1]$ is spanned by the vector $T_{i(B)}$, and therefore $T_1, \dots, T_j$ span the interval $[a_1, b_1 - 1]$. They cannot span a number outside this interval since all of them has either $0^{\{j\}}$ or $1^{\{j\}}$ as their MSBs.

Now to span the first and the third subfunction. Note that there is a one to one correspondence between spanning sets of function $f^n_{[a,a_1-1]}$ and spanning sets of function $f^{n-j+1}_{[a',a_1-1]}$. We get all numbers in the interval of the latter function by removing $0^{\{j-1\}}$ from the beginning of each number in the interval of the former function and vice versa. Similarly there is a one to one correspondence between spanning sets of function $f^n_{[b_1,b]}$ and spanning sets of function $f^{n-j+1}_{[a_1,1b']}$. We get all numbers in the interval of the latter function by removing $1^{\{j-1\}}$ from the beginning of each number in the interval of the former function and vice versa. Let $\mathscr{T}''$ be the set of vectors of length $(n-j+1)$ which spans the function $f^{n-j+1}_{[0a',1b']}$ computed recursively. We use $\mathscr{T}''$ and vectors $T_1, \dots, T_j$ to construct a spanning set of $f^n_{[a,b]}$. We distinguish two cases now.

Case 4.1 ($b' < a' - 1$): The set $\mathscr{T}$ spanning the function $f^n_{[a,b]}$ is the set:

$$\{T_1, \dots, T_j\} \cup \{\phi^{\{j-1\}}T | T \in \mathscr{T}''\} \tag{2.5}$$

As we already know vectors $\{T_i\}_{1 \leq i \leq j}$ span exactly the second subfunction. Let x be the number from $[a, a_1 - 1]$. Then x is spanned by the vector $\phi^{\{j-1\}}T$ where $T \in \mathscr{T}''$ is the vector of length $(n-j)$ spanning the number $x \in [a', a_1 - 1]$. Now let x be the number from $[b_1, b]$. Then x is spanned by the vector $\phi^{\{j-1\}}T$ where $T \in \mathscr{T}''$ is the vector of length $(n-j+1)$ spanning the number $x - (2^n - 2^{n-j+1}) \in [a_1, 1b']$. On the other hand there is no vector in $\mathscr{T}$ spanning some number smaller then a. That would mean that there is a vector spanning the number smaller than $a = a'$ in $\mathscr{T}''$. By the same argument there is no vector in $\mathscr{T}$ spanning some number larger than $b = 1^{\{j\}}b'$.

computeDNF-1int(a,b,n)

Input: Numbers a, b and n such that $0 \le a \le b < 2^n$
Output: Spanning set of function $f^n_{[a,b]}$ of minimum cardinality

if $a = b$ **then return** $\{a\}$
if $a = 0 \wedge b = 2^n - 1$ **then return** $\{\phi^{\{n\}}\}$
if $a^{[1]} = b^{[1]}$ **then**
  $\mathscr{T} :=$ *computeDNF-1int*$(a^{[2,n]}, b^{[2,n]}, n-1)$
  return $\{a^{[1]}T | T \in \mathscr{T}\}$
if $a = 0$ **then**
  $c := b + 1$
  let $o_1, \ldots, o_k$ be the indeces of 1-bits in c
  return $\{c^{[1,o_i-1]}0\phi^{\{n-o_i\}}\}_{i=1}^{k}$
if $b = 2^n - 1$ **then**
  $d := a - 1$
  let $z_1, \ldots, z_k$ be the indeces of 0-bits in d
  return $\{d^{[1,z_i-1]}1\phi^{\{n-z_i\}}\}_{i=1}^{k}$
if $n = 2$ **then return** $\{01, 10\}$
if $a^{[1,2]} = 01 \wedge b^{[1,2]} = 10$ **then**
  $\mathscr{T}_1 =$ *computeDNF-1int*$(a^{[3,n]}, 1^{\{n-2\}}, n-2)$
  $\mathscr{T}_2 =$ *computeDNF-1int*$(0^{\{n-2\}}, b^{[3,n]}, n-2)$
  return $\{01T | T \in \mathscr{T}_1\} \cup \{10T | T \in \mathscr{T}_2\}$
if $a^{[1,2]} = 00 \wedge b^{[1,2]} = 10$ **then**
  $\mathscr{T} =$ *computeDNF-1int*$(a^{[1]}a^{[3,n]}, b^{[1]}b^{[3,n]}, n-1)$
  return $\{01\phi^{\{n-2\}}\} \cup \{T^{[1]}0T^{[2,n-1]} | T \in \mathscr{T}\}$
if $a^{[1,2]} = 01 \wedge b^{[1,2]} = 11$ **then**
  $\mathscr{T} =$ *computeDNF-1int*$(a^{[1]}a^{[3,n]}, b^{[1]}b^{[3,n]}, n-1)$
  return $\{10\phi^{\{n-2\}}\} \cup \{T^{[1]}1T^{[2,n-1]} | T \in \mathscr{T}\}$
if $a^{[1,2]} = 00 \wedge b^{[1,2]} = 11$ **then**
  $j = \max\{i | a^{[1,i]} = 0^{\{i\}} \wedge b^{[1,i]} = 1^{\{i\}}\}$
  $T'_i = \phi^{\{i-1\}}01\phi\{j-1-i\}$ for $i = 1, \ldots, j-1$ and $T'_j = 1\phi^{\{j-2\}}0$
  $\mathscr{T}'' =$ *computeDNF-1int*$(a^{[j,n]}, b^{[j,n]}, n-j+1)$
  if $b^{[j+1,n]} < a^{[j+1,n]} - 1$ **then**
    return$\{T'_1\phi^{\{n-j\}}, \ldots, T'_j\phi^{\{n-j\}}\} \cup \{\phi^{\{j-1\}}T | T \in \mathscr{T}''\}$
  if $b^{[j+1,n]} \ge a^{[j+1,n]} - 1$ **then**
    $\mathscr{T}_1 = \{T'_1\phi^{\{n-j\}}, \ldots, T'_{j-1}\phi^{\{n-j\}}\}$
    $\mathscr{T}_2 = \{\phi^{\{j-1\}}T | T \in \mathscr{T}'' \wedge T^{[1]} \ne 1\}$
    $\mathscr{T}_3 = \{1\phi^{\{j-1\}}T^{\{[2,n-j+1]\}} | T \in \mathscr{T}'' \wedge T^{[1]} = 1\}$
    return $\mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$

Figure 2.4: Construction of minimum spanning set for 1-interval function

Case 4.2 ($b' \leq a' - 1$): In this case define the following set:

$$\begin{aligned}\mathscr{T}' =& \{\phi^{\{j-1\}}T | T \in \mathscr{T}'' \wedge T^{[1]} \in \{0, \phi\}\} \\ & \cup \{1\phi^{\{j-1\}}T^{[2,n-j+1]} | T \in \mathscr{T}'' \wedge T^{[1]} = 1\}\end{aligned} \tag{2.6}$$

The set $\mathscr{T}$ spanning the interval $[a, b]$ is the set $\mathscr{T}' \cup \{T_1, \ldots, T_{j-1}\}$. Note that we have leaved out vector $T_j = 1\phi^{\{j-2\}}0\phi^{\{n-j\}}$. To see that $\mathscr{T}$ spans exactly the function $f^n_{[a,b]}$ note that vectors $T_1, \ldots, T_{j-1}$ span all numbers whose j MSB's contain at lest one 0-bit and at least one 1-bit except those spanned exclusively by T_j. Let J denote this set of unspanned numbers. That is all n-bit numbers whose j MSB's are $1^{\{i\}}0^{\{j-i\}}$ for some $1 \leq i < j$. Now we need to span numbers from J and subfunctions $f^n_{[a,a_1-1]}$ and $f^n_{[b_1,b]}$. We claim that all of these are spanned by $\mathscr{T}'$. By the same argument as in case 4.1 $\mathscr{T}'$ spans all numbers from intervals $[a, a_1 - 1]$ and $[b_1, b]$. To see that $\mathscr{T}'$ spans all numbers in J suppose x' is any $(n - j)$-bit number. If $x' \geq a'$ then $\mathscr{T}''$ spans the $(n - j + 1)$-bit number $0x'$. If $x' < a'$ then $x' \leq b'$ and $\mathscr{T}''$ spans the $(n - j + 1)$-bit number $1x'$. In both cases, the corresponding vector of $\mathscr{T}'$ spans all n-bit numbers of the form $1^{\{i\}}0^{\{j-i\}}x'$, for $1 \leq i < j$. Therefore $\mathscr{T}'$ spans all numbers in J.

There is a small difference between algorithm on figure as it is written here and as it is presented in [2]. It is on line 14 where we added the solution for case $n = 2$. The original form of algorithm was not complete. If we want to span a function $f^2_{[1,2]}$ it will result in recursive call to interval defined by empty vectors which is undefined. At first it does not seem as a mistake since in [2] algorithm is presented with assumption of $n \geq 3$. But the spanning set of $f^2_{[1,2]}$ is needed during computation of spanning set of f^n for $n \geq 3$ as well. Take $f^3_{[1,6]}$ as an example.

Theorem 2.1. *Spanning set generated by the algorithm in figure 2.4 is of minimum cardinality.*

Proof. We prove the theorem by induction. The case $n \leq 2$ is trivial. Assume that the theorem holds for $n' < n$. Let $\mathscr{T}$ be any spanning set of $f^n_{[a,b]}$. We prove each case separately.

Case 1: We have $a^{[1,2]} = 01$ and $b^{[1,2]} = 10$. There is no vector spanning some part of $f^n_{[a,2^{n-1}-1]}$ and some part of $f^n_{[2^{n-1},b]}$ at the same time. That would imply $T^{[1,2]} = \phi\phi$ but that would mean that T also spans some number with 11 as its two MSB's. But such a number is greater then b. So we can see that we can divide each spanning set of $f^n_{[a,b]}$ into two disjount spanning sets of $f^n_{[a,2^{n-1}-1]}$ and $f^n_{[2^{n-1},b]}$ respectively. Therefore it is enough to find the minimum spannig set of suffix subfunction $f^n_{[a,2^{n-1}-1]}$ and the minimum spanning set of the prefix subfunction $f^n_{[2^{n-1},b]}$. Their union is a minimum spanning set of $f^n_{[a,b]}$. This is what our algorithm does.

Case 2: We have $a^{[1,2]} = 00$ and $b^{[1,2]} = 10$. Using $\mathscr{T}$ we are going to construct set $\mathscr{T}'$ which spans exactly those numbers from interval $[a, b]$ whose MSBs are either 00 or 10. We shall go through all vectors in $\mathscr{T}$ by their two MSBs.

$T^{[1,2]}$ **equals to one of these:** 11, 1ϕ, $\phi1$, $\phi\phi$

There is no such a vector in $\mathscr{T}$ because it would span a number with 11 as its two MSBs. And such a number is a falsepoint because it is greater then b.

$T^{[1,2]}$ **equals to one of these:** 00**,** 10**,** $\phi 0$
These vectors span only numbers having 00 or 10 as their MSBs exactly as we want. We put it into $\mathscr{T}'$ without modification.

$T^{[1,2]}$ **equals to** 01
Such a vector does not span any number we want so we can leave it out during construction of $\mathscr{T}'$.

$T^{[1,2]}$ **equals to** 0ϕ
Such a vector span both 00 which we want and 01 which we do not. We can easily modificate this vector by fixing its second bit $T^{[1,2]} = 00$.

So we construct desired set as follows:

$$\mathscr{T}' = \{T | T \in \mathscr{T} \wedge T^{[2]} = 0\} \cup \{T^{[1]} 0 T^{[3,n]} | T \in \mathscr{T} \wedge T^{[1,2]} = 0\phi\} \tag{2.7}$$

It is not hard to verify that $\mathscr{T}'$ spans exactly truepoints of $f^n_{[a,b]}$ having 00 or 10 as their MSBs. Note that during construction we have leaved out vectors spanning only numbers with 01 as their MSBs. We are going to show that there had to be at least one such a vector in $\mathscr{T}$.

Consider vector $T \in \mathscr{T}$ spanning number $010^{\{n-2\}}$. It has to be that $T^{[1]} = 0$ otherwise T would span number $b < 110^{\{n-2\}}$ which is falsepoint. Moreover, it has to be that $T^{[1,2]} = 01$ becase otherwise T would span number $0^{\{n\}}$ which is again a falsepoint. So we have found a vector spanning only numbers with 01 as its first two bits. Therefore we have $|\mathscr{T}'| + 1 \leq |\mathscr{T}|$. Moreover $\mathscr{T}'$ is at least the minimum size of a set spanning function $f^n_{[a,2^{n-2}-1],[2^{n-1},b]}$ which is in turn at least the minimum size spanning the function $f^{n-1}_{[a',b]}$ where $a' = a^{[1]} a^{[3,n]}$ and $b' = b^{[1]} b^{[3,n]}$.

So we have shown lower bound for the size of spanning set of $f^n_{[a,b]}$. By induction hypothesis this is the size of set constructed by our algorithm.

Case 4: We have $a^{[1,2]} = 00$ and $b^{[1,2]} = 11$. Again let $j \geq 2$ be the maximum number such that $a^{[1,j]} = 0^{\{j\}}$ and $b^{[1,j]} = 1^{\{j\}}$. Let $a' = a^{[j,n]}$ and $b' = b^{[j,n]}$.

Case 4.1: $b' < a' - 1$. We will show that the vectors in $\mathscr{T}$ can be partitioned into two subsets. One of size at least j spanning subfunction $f^n_{[a_1,b_1-1]}$ and the other spanning subfunctions $f^n_{[a,a_1-1]}$ and $f^n_{[b_1,b]}$. Choose any $(n-j)$-bit number c such that $b' < c < a'$. Define j n-bit numbers $e_1, \ldots, e_j$ as all cyclic shifts of vector $10^{\{j-1\}}$ concatenated with number c. It is easy to see that for every e_i

$$e_i \in [a_1, b_1 - 1] \subseteq [a, b] \tag{2.8}$$

And so they are truepoints of $f^n_{[a,b]}$. We claim that $\mathscr{T}$ contains at least j vectors which span only truepoints from subfunction $f^n_{[a_1,b_1-1]}$. To see this, first note that no vector in $\mathscr{T}$ can span any pair of truepoints e_x, e_y $(x \neq y)$. Indeed, any ternary vector that spans a pair e_x, e_y has a ϕ in positions x and y and either 0 or ϕ in the remaining j most significant positions, and therefore also spans a falsepoint $c < a$. Note also that no vector in $\mathscr{T}$ spans a truepoint e_x and some truepoint of $f^n_{[a,a_1-1]}$ or $f^n_{[b_1,b]}$. Again, this is because any vector that spans both e_x and a truepoint of $f^n_{[a,a_1-1]}$, also spans the falsepoint $c < a$. Similarly, any vector that spans both e_x and a truepoint of the $f^n_{[b_1,b]}$, also spans the falsepoint $b_1 + c > b$. In both cases such vectors cannot belong to $\mathscr{T}$.

Recall that there is a one to one correspondence between spanning sets of a function $f^n_{[a,a_1-1]}$ and spanning sets of s function $f^{n-j+1}_{[a'',a_1-1]}$ where $a'' = a^{[j,n]}$. There is also a one to one correspondence between spanning sets of a function $f^n_{[b_1,b]}$ and spanning sets of a function $f^{n-j+1}_{[2^{n-j},b'']}$ where $b'' = b^{[j,n]}$. We conclude that the size of $\mathscr{T}$ is at least j plus the minimum size of the set spanning a function $f^{n-j+1}_{[a'',b'']}$. By induction hypothesis this is the size of the set computed by the algorithm.

Case 4.2: $b' \geq a' - 1$. Consider the MSBs of a' and b'. By the definition of j it cannot be that $a'^{[1]} = 0$ and $b'^{[1]} = 1$. Suppose that $a'^{[1]} = 1$ and $b'^{[1]} = 0$. Since $b' \geq a' - 1$ this can happen only if $b' = 01^{\{n-j-1\}}$ and $a' = 10^{\{n-j-1\}}$. Observe that in this case our algorithm generates a spanning set of size $j + 1$. These are the vectors $T_1, \ldots, T_{j-1}$, the vector $\phi^{\{j-1\}}01\phi^{\{n-j-1\}}$ and the vector $1\phi^{\{j-1\}}0\phi^{\{n-j-1\}}$. Define $j + 1$ n-bit numbers $e_1, \ldots, e_{j+1}$ given by concatenating every cyclic shift of the vector $10^{\{j\}}$ of the length $j + 1$ with $0^{\{n-j-1\}}$. Clearly, all these numbers are in the interal $[a, b]$ and thus they are truepoints. Similarly as in Case 4.1., no two numbers e_x and e_y can be spanned by the same vector in $\mathscr{T}$. So truepoints $e_1, \ldots, e_{j+1}$ forms an orthogonal set for $f^n_{[a,b]}$. By observation 1.1 this is a lower bound and it shows that the set constructed by the algorithm is minimal.

We are left with two complementing cases: either both MSBs are 0-bits or both are 1-bits. We consider the case of 0-bits, the other case is symmetric. We will show that $\mathscr{T}$ contains two disjoint subsets of vectors. One of size at least j that does not span any number in the two sub-intervals $[a, 2^{n-j-1} - 1]$ and $[b_1, b]$, and the other that spans the two subintervals $[a, 2^{n-j-1} - 1]$ and $[b_1, b]$ and possibly numbers in $[2^{n-j-1}, b_1 - 1]$ as well.

Now we shall construct an orthogonal set $\bar{e}_1, \ldots, \bar{e}_j$ for $f^n_{[a,b]}$ such that every $\bar{e}_i \in [2^{n-j-1}, b_1 - 1]$. They are given by concatenating every cyclic shift of the vector $01^{\{j-1\}}$ of length j with 10^{n-j-1}. As in Case 4.1, no vector in $\mathscr{T}$ spans a pair $\bar{e}_x, \bar{e}_y$ $(x \neq y)$. This is because any vector that spans both $\bar{e}_x$ and $\bar{e}_y$ must have a ϕ in positions x and y, either 1 or ϕ in the remaining most significant $j+1$ positions and either 0 or ϕ in the $n - j - 1$ least significant positions. However, such a vector also spans the falsepoint $1^{\{j+1\}}0^{\{n-j-1\}}$, which is greater than b. Moreover, no vector in $\mathscr{T}$ spans a truepoint $\bar{e}_x$ and a number in the subinterval $[b_1, b]$, because this would again imply that the vector spans a falsepoint greater than b. We claim that no vector in $\mathscr{T}$ spans a truepoint $\bar{e}_x$ and a number in the subinterval $[a, 2^{n-j-1} - 1]$. This is because any vector spanning such a pair must have a ϕ-bit in position $j + 1$, and either 0-bit or ϕ-bit in the remaining positions. However, such a vector also spans the number $0 < a$. We conlude that $\mathscr{T}$ contains at least j vectors that do not span any number in the two subintervals $[a, 2^{n-j-1} - 1]$ and $[b_1, b]$.

Let $\mathscr{S}$ be the minimum set of vectors in $\mathscr{T}$ that spans two subintervals $[a, 2^{n-j-1} - 1]$ and $[b_1, b]$. Notice that $\mathscr{S}$ may also span numbers in $[2^{n-j-1}, b_1 - 1]$. From the minimality of $\mathscr{S}$ it follows that the symbol in position $(j + 1)$ if each vector in $\mathscr{S}$ is either a 0 or a ϕ. It also follows that no vector in $\mathscr{S}$ has both a 0-bit and a 1-bit together in its j most significant symbols. Therefore, the set $\mathscr{S}$ induces a set $\mathscr{S}'$ (of smaller or equal size) that spans exactly the two subintervals $[a'', 2^{n-j-1}-1]$ and $[2^{n-j}, b'']$ (and no other number) as follows. For each vector $S \in \mathscr{S}$, if $S^{[1,j]} = \phi^{\{j\}}$ then the corresponding vector in $\mathscr{S}'$ is $\phi 0 S^{[j+2,n]}$. Otherwise, the corresponding vector in $\mathscr{S}'$ is $00S^{[j+2,n]}$, if $S^{[1,j]}$ contains a 0, and $10S^{[j+2,n]}$, if $S^{[1,j]}$ contains a 1. Clearly, $\mathscr{S}'$ does not span any number in $[2^{n-j-1}, a_1 - 1]$. In addition, no vector $S' \in \mathscr{S}'$ spans any number smaller than a'' or greater than b''. That would imply that S spans a number smaller than a or greater then b, respectively. Recall that there is a one to one correspondance between spanning sets of a function $f^n_{[a,2^{n-j-1}-1]}$ and spanning sets of a function $f^{n-j+1}_{[a'',2^{n-j-1}-1]}$. And there is a one to one correspondance between spanning sets of a function $f^n_{[b_1,b]}$ and spanning sets of a function $f^{n-j+1}_{[2^{n-j},b'']}$. We conclude that the size of $\mathscr{T}$ is at least j plus the minimum size of a set spanning the two subinterval $[a'', 2^{n-j-1}]$ and $[2^{n-j}, b'']$.

We now show that this lower bound is achieved by our algorithm. Recall that the spanning set generated by our algorithm (call it $\mathscr{T}_a$) contains the $j-1$ vectors $T_1, \ldots, T_{j-1}$ which do not span any number in the two sub-intervals $[a, 2^{n-j}-1]$ and $[b_1, b]$. The remaining vectors of $\mathscr{T}_a$ are obtained by recursively computing a spanning set for the interval $[a'', b'']$. Note that $a''^{[1,2]} = 00$ and $b''^{[1,2]} = 10$. Applying case 2 it follows that the computed spanning set of $[a'', b'']$ consists of the vector $01\phi^{\{n-j-1\}}$ and the minimum size set of vectors that spans the two subintervals $[a'', 2^{n-j-1}-1]$ and $[2^{n-j}, b'']$. The vector in $\mathscr{T}_a$ that corresponds to the $(n-j+1)$-bit vector $01\phi^{\{n-j-1\}}$ is $\phi^{\{j-1\}}01\phi^{\{n-j-1\}}$. Clearly, this vector does not span any number in the two subintervals $[a, 2^{n-j-1}-1]$ and $[b_1, b]$. We conclude that the size of $\mathscr{T}_a$ is equal to j plus the minimum size of a set spanning the two subintervals $[a'', 2^{n-j-1}-1]$ and $[2^{n-j}, b'']$, as desired. $\square$

The presented proof has an interesting property. Algorithm itself produces a spanning set. This proof of minimality enables us to modificate algorithm in such a way that it would provide not only spanning set but an orthogonal set of of the same size as well. That is the idea behind the proof. We immediately have following observation.

Observation 2.1. *For all functions $f^n_{[a,b]}$ the size of its minimum spanning set is the same as the size of maximum orthogonal set given $f^n_{[a,b]}$.*

This is a solution to the hypothesis 1.1 for class of 1-interval functions.

3. Spanning sets of functions of type A

In this chapter we can finally turn our attention to 2-interval functions themselves. Here in this section we will deal with functions of first type i.e. those in form of $f^n_{[0,b],[c,2^n-1]}$. Note that function which is identically true is member of this class. We start by determining the number of functions of n variables in this class.

The function is fully defined by pair of numbers b and c. It is because a and d are fixed in their values. So the number of functions is a binomial coefficient. But a little correction is needed. Function $f^n_{[0,b],[b+1,2^n-1]}$ is identically true for all $2^n - 1$ possible values of b. We can now state that:

$$|A_0| - 1 = \binom{2^n}{2} - (2^n - 1) \tag{3.1}$$

We excluded identically true function by minus one on left side. On right side we have the number of pairs minus the number of cases where $b + 1 = c$. So we can write:

$$\begin{align} |A_0| &= \binom{2^n}{2} - (2^n - 1) + 1 \tag{3.2} \\ &= \frac{(2^n)!}{2(2^n - 2)!} - (2^n - 1) + 1 \tag{3.3} \\ &= 2^{n-1}(2^n - 1) - (2^n - 1) + 1 \tag{3.4} \\ &= (2^n - 1)(2^{n-1} - 1) + 1 \tag{3.5} \end{align}$$

3.1 Approximation

At first we can think about how we can reuse an optimization algorithm for 1-interval functions. Having two intervals to be spanned it is reasonable to try to span each of them separately by algorithm in figure 2.4. Union of two resulting spanning sets needs to be a spanning set of two intervals. Moreover it is natural to ask how far this solution is from an optimal one. Lemma 3.1 formalizes these ideas.

Lemma 3.1. *(Approximation) Let $f^n_{[0,b],[c,2^n-1]}$ be function which is not uniformly true. Let $\mathscr{T}_1$ be the minimum spanning set of function $f^n_{[0,b]}$ and let $\mathscr{T}_2$ be the minimum spanning set of function $f^n_{[c,2^n-1]}$. Moreover let $\mathscr{T}$ be the minimum spanning set of function $f^n_{[0,b],[c,2^n-1]}$. Than it holds that $|\mathscr{T}_1 \cup \mathscr{T}_2| < 2|\mathscr{T}|$.*

Proof. Without loss of generality we assume that $|\mathscr{T}_2| \leq |\mathscr{T}_1|$. We start by proving that $|\mathscr{T}_1| \leq |\mathscr{T}|$. Denote by S the orthogonal set for function $f^n_{[0,b]}$ constructed as (2.1) in lemma 2.1. From the proof of lemma we know that its size is the same as the size of the minimum spanning set of $f^n_{[0,b]}$. As a result we have $|S| = |\mathscr{T}_1|$. We claim that S is an orthogonal set given $f^n_{[0,b],[c,2^n-1]}$. The proof of ortogonality of S for function $f^n_{[0,b]}$ relies only on one falsepoint of that function. Namely number

approxDNF-typeA(b,c,n)

Input: Numbers b, c and n such that $0 \leq b < c < 2^n$
Output: Spanning set of function $f^n_{[0,b],[c,2^n-1]}$

1: $\mathscr{T}_1 :=$ *computeDNF*$(0,\ b,\ n)$
2: $\mathscr{T}_2 :=$ *computeDNF*$(c,\ 2^n-1,\ n)$
3: **return** $\mathscr{T}_1 \cup \mathscr{T}_2$

Figure 3.1: Approximation of minimum spanning set for 2-interval function of type A

$b+1$. Since $f^n_{[0,b],[c,2^n-1]}$ is not uniformly true we see that $b+1$ is a falsepoint of the 2-interval function as well. Therefore the proof of orthogonality of S for $f^n_{[0,b]}$ is valid also as a proof of orthogonality of S for $f^n_{[0,b],[c,2^n-1]}$. Now by observation(1.1) we have $|\mathscr{T}_1| = |S| \leq |\mathscr{T}|$.

By $|\mathscr{T}_2| \leq |\mathscr{T}_1|$ we now have $|\mathscr{T}_1 \cup \mathscr{T}_2| \leq 2|\mathscr{T}|$. We just need to show this inequality to be strict. Trying to reach a contradiction we assume an equality holds. Using proved inequality $|\mathscr{T}_2| \leq |\mathscr{T}_1| \leq |\mathscr{T}|$ and the fact that $\mathscr{T}_1$ and $\mathscr{T}_2$ are disjoint it is easy to show that it implies that $|\mathscr{T}_1| = |\mathscr{T}|$. Denote this size by k_1. Now we claim that the set $S \cup \{1^{\{n\}}\}$ is an orthogonal set as well. S itself is an orthogonal set and $1^{\{n\}}$ is a truepoint. Moreover, let T be a ternary vector spanning truepoint $1^{\{n\}}$ and some $T_x \in S$. By the construction of S we know that vector T_x was furnished from number $b+1$ by turining one of its 1-bits into 0-bit. Let $T_x^{[i]} = 0$ be an index of that bit. However since T spans $1^{\{n\}}$ we know that $T^{[i]} = \phi$. Moreover, at every index except i number T_x is identical to $b+1$. Therefore vector T has to span falsepoint $b+1$ as well. That means that $S \cup \{1^{\{n\}}\}$ is an orthogonal set for f of size k_1+1. By observation(1.1) we have that $k_1 + 1 \leq |\mathscr{T}|$. But that contradicts the fact that $k_1 = |\mathscr{T}_1| = |\mathscr{T}|$. Thus it is not possible for equality to hold. ☐

We have proven that this simple solution provides 2-approximate algorithm. Later in the text we will be able to show that this bound for error is tight.

3.2 Optimization algorithm

Now we are ready to present an optimization algorithm for 2-interval functions of type A. Inputs of algorithm are numbers b and c and number of variables n such that $b < c < 2^n$. Output is a minimum spanning set of function $f^n_{[0,b],[c,2^n-1]}$. We assume that the function is not uniformly true. Otherwise the minimum spanning set is indeed $\{\phi^{\{n\}}\}$. The algorithm distinguishes seven cases. Description of case number i starts by definition of subclass A_i of functions solved by the case. Then the solution and the proof of the minimality follows. Note that classes $A_1, \ldots, A_7$ are disjoint and that their union is A.

Case 1 deals with functions of class $A1 \subset A$ with following properties:

1. There is at least one falsepoint for f
2. $b + 1 < 2^n - c$

This is just technical case which ensures for other cases that the interval $[0, b]$ is at least as long as the interval $[c, 2^n - 1]$. Every function not fulfiling this request falls here and is simply mirrored before next processing.

Lemma 3.2. *(Case A1) Let f^n be a function from class A_1. Let $\mathscr{T}'$ be a minimum spanning set of function $g^n_{[0,\bar{c}],[\bar{b},2^n-1]}$. Then $\mathscr{T} = \{comp(T) | T \in \mathscr{T}'\}$ is a minimum spanning set of f^n. Moreover function g^n does not belong to class A_1.*

Proof. We can see that $x < y$ iff $\bar{y} < \bar{x}$. That means that g^n is correctly defined function. Then set $\mathscr{T}$ is indeed minimum spanning set of f^n. We just need to show that g^n is outside of $A1$. Since f^n is from $A1$ we know that $b + 1 < 2^n - c$. After using the identity $x = 2^n - 1 - \bar{x}$ we have $\bar{c} + 1 > 2^n - \bar{b}$. □

Case 2: Let A_2 be the subclass of $A \setminus A_1$ containing functions $f^n_{[0,b],[c,2^n-1]}$ with following properties:

1. there is at least one falsepoint of f
2. $b + 1 \geq 2^n - c$
3. $01^{\{n-1\}}$ is a truepoint

Conditions number 1 and 3 togather implies that there is no falsepoint among the leaves in left subtree. We can show that it is spanned by standalone ternary vector in some minimum spanning set.

Lemma 3.3. *(Case A2) Let f be a function from class A_2. Let $\mathscr{T}$ be the minimum spanning set of the restriction of f on the subinterval $[2^{n-1}, 2^n - 1]$. Then the set $\mathscr{T} \cup \{0\phi^{\{n-1\}}\}$ is minimum spanning set of f.*

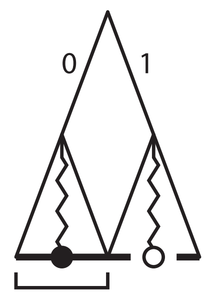

Figure 3.2: Construction of member of orthogonal set in case 2

Proof. From properties of definition of A_2 we can see that neccesarily there exists some falsepoint in interval $[2^{n-1}, 2^n - 1]$. Denote it by e. It means that $e^{[1]} = 1$. Let $\mathscr{T}'$ be any spanning set of function f. Let $T \in \mathscr{T}'$ be any vector spanning truepoint $0e^{[2,n]}$. Since T cannot cover falsepoint e we now see that $T^{[1]} = 0$. That means that $\mathscr{T}' \setminus \{T\}$ is a spanning set which spans every truepoint in the subinterval $[2^{n-1}, 2^n - 1]$. And thus, by the minimality of $\mathscr{T}$, we have $|\mathscr{T}| \leq |\mathscr{T}' \setminus \{T\}| = |\mathscr{T}'| - 1$. Therefore we have $|\mathscr{T} \cup \{0\phi^{\{n-1\}}\}| = |\mathscr{T}| + 1 \leq |\mathscr{T}'|$ proving the minimality. □

The construction of a minimum spanning set of function f^n which satisfies the assumptions of lemma 3.3 is reduced to a solution of function $f^{n-1}_{[0,b'],[c',2^{n-1}]}$ where $b' = b - 2^{n-1}$ and $c' = c - 2^{n-1}$. We got this function as $f[x_1 := 1]$. If $b = 2^{n-1} - 1$ than $f[x_1 := 1]$ is 1-interval function and can be solved by algorithm in figure 2.4. Otherwise $f[x_1 := 1]$ is again 2-interval function in the class A and can be solved recursively.

Case 3: Let A_3 be the subclass of $A \backslash (A_1 \cup A_2)$ containing functions $f^n_{[0,b],[c,2^n-1]}$ with following properties:

1. $b + 1 \geq 2^n - c$
2. $01^{\{n-1\}}$ is a falsepoint
3. $b^{[2,n]} < c^{[2,n]}$

Lemma 3.4. *(Case A3) Let f^n be a function from class A_3. Let $\mathscr{T}_1$ be the minimum spanning set of 1-interval function $f^n_{[0,b]}$. Let $\mathscr{T}_2$ be the minimum spanning set of 1-interval function $f^n_{[c,2^n-1]}$. Then the set $\mathscr{T}_1 \cup \mathscr{T}_2$ is the minimum spanning set of $f^n_{[0,b],[c,2^n-1]}$.*

Proof. We know that $b^{[2,n]} < c^{[2,n]}$. We shall see that this assumption implies that every vector having ϕ as its first bit spans at least one falsepoint. Let T be the vector such that $T^{[1]} = \phi$ and let x be any number spanned by T. If x is falsepoint we are done. Without loss of generality assume that x is truepoint with $x^{[1]} = 0$. That means that $x \leq b$. Note that $b^{[1]} = 0$ because of the assumption of $01^{\{n-1\}}$ being a falsepoint. This implies that $x^{[2,n]} \leq b^{[2,n]} < c^{[2,n]}$. Since also $c^{[1]} = 1$ we have $b < 1x^{[2,n]} < c$. This means that number $1x^{[2,n]}$ is falsepoint. Since vector T spans truepoint x and $T^{[1]} = \phi$ then T spans also falsepoint $1x^{[2,n]}$.

Let e_i for $1 \ldots k_1$ be orthogonal set given $f^n_{[0,b]}$ constructed by lemma 2.1. Similarly let y_i for $1 \ldots k_2$ be orthogonal set given $f^n_{[c,2^n-1]}$ constructed by lemma 2.2. We show that union of theese two sets is orthogonal set given $f^n_{[0,b],[c,2^n-1]}$. Only we need to show is that pair of truepoints from different orthogonal sets cannot be covered by one vector without covering some falsepoint as well. Now consider such a pair of points e_i and y_j. It holds that $e_i^{[1]} = 0$ and $y_j^{[1]} = 1$. The pair cannot be spanned by one vector because assumption $b^{[2,n]} < c^{[2,n]}$ ensures that there is no ternary vector of length n with ϕ as its first bit covering only truepoints.

We constructed orthogonal set given $f^n_{[0,b],[c,2^n-1]}$. By observation (1.1) number $k_1 + k_2$ is lower bound on the size of minimum spanning set of f. By the construction of sets e_i and y_i by prefix and suffix lemmas we know that $|\mathscr{T}_1| = k_1$ and $|\mathscr{T}_2| = k_2$. Therefore $\mathscr{T}_1 \cup \mathscr{T}_2$ is indeed a spanning set of minimum size. $\square$

Case 4: Let A_4 be the subclass of $A \setminus (A_1 \cup A_2 \cup A_3)$ containing functions $f^n_{[0,b],[c,2^n-1]}$ with following properties:

1. $b + 1 \geq 2^n - c$
2. $01^{\{n-1\}}$ is a falsepoint
3. $b^{[2,n]} \geq c^{[2,n]}$
4. $101^{\{n-2\}}$ is a falsepoint

Conditions 2 and 4 together means that there is no falsepoint in interval $[2^{n-1}, 2^{n-1}+2^{n-2}-1]$. Conditions 3 and 4 togather implies that number $001^{\{n-2\}}$ is a truepoint and thus there is no falsepoint in the interval $[0, 2^{n-2}-1]$. By next lemma we show that there is a spanning set of minimum size where this interval is spanned by standalone ternary vector.

Lemma 3.5. *(Case A4) Let f^n be the function from class A_4. Let $\mathscr{T}_r$ be the minimum spanning set of restricted function $g^{n-1} = f[x_2 := 1]$. Then the set of vectors $\{v^{[1]}1v^{[2,n-1]} | v \in \mathscr{T}_r\} \cup \{00\phi^{\{n-2\}}\}$ is spanning set of f^n of minimum cardinality.*

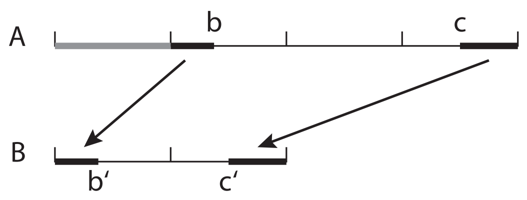

Figure 3.3: Construction of function to be solved recursively in case 4

Proof. First we shall see that constructed set is a spanning set. All numbers having 00 as their first two bits are truepoints. They are spanned by vector $00\phi^{\{n-2\}}$. All numbers having 10 as their first two bits are falsepoints. Obviously there is no vector in our set which would span some number begining with 10 which is correct. All remaining numbers are spanned correctly because our assumption of $\mathscr{T}_r$ being a minimum spanning set of restricted function $f[x_2 := 1]$.

We proceed by proving the minimality. Let $\mathscr{T}$ be any spanning set of f. We can see that the following set is the spanning set of $f[x_2 := 1]$:

$$\mathscr{T}' = \{T^{[1]}T^{[3,n]} | T \in \mathscr{T} \wedge T^{[2]} \in \{1, \phi\}\} \tag{3.6}$$

Denote by $\mathscr{T}^{n-1}_{min}$ spanning set of minimum size of $f^{n-1}_{[0,b'],[c',2^{n-1}-1]}$ where $b' = b - 2^{n-2}$ and $c' = c - 2^{n-1}$. There is one to one correspondence between spanning sets of $f^{n-1}_{[0,b'],[c',2^{n-1}-1]}$ and spanning sets of $f[x_2 := 1]$. Thus we have $|\mathscr{T}_r| = |\mathscr{T}^{n-1}_{min}| \leq |\mathscr{T}'|$.

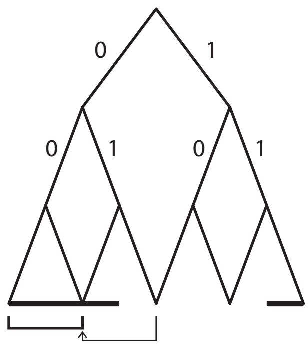

Figure 3.4: Construction of member of orthogonal set in case 4

Now we consider number $001^{\{n-2\}}$ and vector $T \in \mathscr{T}$ spanning it. It has to be the case that $T^{[2]} = 0$. Otherwise T would also span falsepoint $01^{\{n-1\}}$. That implies $T \notin \mathscr{T}'$. Now we have shown that $|\mathscr{T}'| + 1 \leq |\mathscr{T}|$. Summarizing inequalities we have proved we have $|\mathscr{T}^{n-1}_{min}| + 1 = |\mathscr{T}^{n}_{min}|$ where by $\mathscr{T}^{n}_{min}$ we denote spanning set of f of minimum size. As we can see the spanning set we presented in the statement of the lemma has this size. $\square$

Case 5: Let A_5 be the subclass of $A \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$ containing functions $f^n_{[0,b],[c,2^n-1]}$ with following properties:

1. $b + 1 \geq 2^n - c$
2. $01^{\{n-1\}}$ is a falsepoint
3. $101^{\{n-2\}}$ is a truepoint
4. $b^{[3,n]} + 1 < c^{[3,n]}$

Note that we have leaved out condition $b^{[2,n]} \geq c^{[2,n]}$. It is because from first three conditions we see that number b is from $[2^{n-2}, 2^{n-1} - 1]$ and number c is from $[2^{n-1}, 2^{n-1} + 2^{n-2} - 1]$. In such circumstances condition $b^{[2,n]} \geq c^{[2,n]}$ is equivalent to $b + 1 \geq 2^n - c$. Now when we know that b and c belong to above mentioned intervals we know what condition number 4 means. It says that there is a falsepoint $01x$ such that $10x$ is falsepoint as well (see figure 3.5).

Lemma 3.6. *(Case A5) Let f^n be the function from class A_5. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be the minimum spanning sets of $f^n_{[0,b]}$ and $f^n_{[c,2^n-1]}$ respectively. Then $\mathscr{T}_1 \cup \mathscr{T}_2$ is minimum spanning set of f^n.*

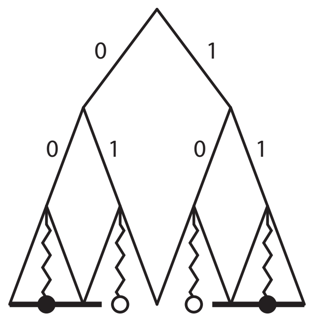

Figure 3.5: Two members of an orthogonal set constructed from falsepoints in case 5

Proof. Let $E_1, \ldots, E_{k_1}$ be the orthogonal set for $f^n_{[0,b]}$ constructed in lemma 2.1. Note that due to the construction (2.1) vector E_1 is in $[0, 2^{n-2} - 1]$. Similarly let $Y_1, \ldots, Y_{k_2}$ be the orthogonal set for $f^n_{[c,2^n-1]}$ described in lemma 2.2 where Y_1 is the number from $[2^n - 2^{n-2}, 2^n - 1]$. From proofs of lemmas 2.1 and 2.2 we know that $k_1 = |\mathscr{T}_1|$ and $k_2 = |\mathscr{T}_2|$. Since $b^{[3,n]} + 1 < c^{[3,n]}$ holds there has to be some vector of length $n-2$ representing the number greater than $b^{[3,n]}$ and smaller then $c^{[3,n]}$. Let x be such a vector. Notice that this choice implies that both $01x$ and $10x$ are falsepoints. Now we define the set:

$$\mathscr{E} = \{00x\} \cup \{E_2, \dots, E_{k_1}\} \cup \{11x\} \cup \{Y_2, \dots, Y_{k_2}\} \tag{3.7}$$

The set $\mathscr{T}_1 \cup \mathscr{T}_2$ is evidently a spanning set of f^n of size $k_1 + k_2$. It is easy to see that $\mathscr{E}$ has the same size. If we want to prove that $\mathscr{T}_1 \cup \mathscr{T}_2$ is a minimum spanning set then by lemma 1.2 it is enough to show that $\mathscr{E}$ is an orthogonal set for f^n.

At first we see that all vectors from $\mathscr{E}$ are truepoints. Number b has 01 as its two MSBs so $00x$ is in interval $[0, b]$. Number c has 10 as its two MSBs so $11x$ is in interval $[c, 2^n - 1]$. All vectors E_i belongs to interval $[0, b]$ by their construction. By the same reason all vectors Y_i belongs to interval $[c, 2^n - 1]$. So all vectors in $\mathscr{E}$ are truepoints.

Now we need to show that all pairs of vectors T_1 and T_2 from $\mathscr{E}$ are orthogonal. We distinguish this five cases:

1. $T_1 = 00x \wedge T_2 \in \{E_2, \dots, E_{k_1}\}$
2. $T_1 = 00x \wedge T_2 \in \{Y_2, \dots, Y_{k_2}\}$
3. $T_1 = 00x \wedge T_2 = 11x$
4. $T_1, T_2 \in \{E_2, \dots, E_{k_1}\}$
5. $T_1 \in \{E_2, \dots, E_{k_1}\} \wedge T_2 \in \{Y_2, \dots, Y_{k_2}\}$

The rest of cases are symmetric. Now let T be a ternary vector spanning both T_1 and T_2. We are going to show that T has to span some falsepoint as well.

Case 1: In this case $T_1^{[1,2]} = 00$ and $T_2^{[1,2]} = 01$. Second equality holds by construction of $\{E_1, \dots, E_{k_1}\}$. That implies that $T^{[2]} = \phi$. So T also spans falsepoint $01x$.

Case 2: In this case we have $T_1^{[3,n]} = x$ and $T_2^{[1,2]} = 10$. Second equality again holds by construction of $\{Y_1, \dots, Y_{k_2}\}$. So T spans falsepoint $10x$ as well.

Case 3: From the form of T_1 and T_2 we see that $T^{[1,2]} = \phi\phi$ and so T spans also falsepoint $01x$.

Case 4: Both vectors come from lemma 2.1. In its proof it's shown that each vector spanning both of them spans also vector $b + 1$ which is a falsepoint. And so does T.

Case 5: We have the following inequalities. They hold by construction of x.

$$T_1 < T_1^{[1,2]}x < T_1^{[1,2]}T_2^{[3,n]} < T_2^{[1,2]}x < T_2 \tag{3.8}$$

Vectors $T_1^{[1,2]}x = 01x$ and $T_2^{[1,2]}x = 10x$ are falsepoints. That is how we defined x. But that means that $T_1^{[1,2]}T_2^{[3,n]}$ is falsepoint as well. Moreover it is easy to see that $T_1^{[1,2]}T_2^{[3,n]}$ is spanned by vector T as well. $\square$

Definition 3.1 (Outer point)**.** *Let $\mathscr{E}$ be an orthogonal set for function f^n. We say that $e \in \mathscr{E}$ is an outer point of $\mathscr{E}$ if one of the following holds:*

1. $e^{[1,2]} = 00 \wedge 01e^{[3,n]}$ *is a truepoint of* f^n
2. $e^{[1,2]} = 11 \wedge 10e^{[3,n]}$ *is a truepoint of* f^n

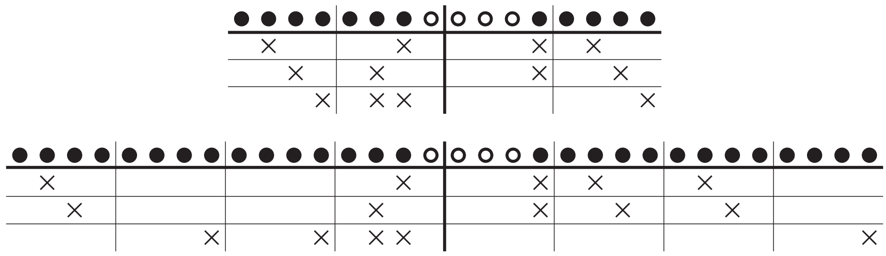

Figure 3.6: Case 6: All orthogonal sets for $f^4_{[0,6],[11,15]}$ and for $f^5_{[0,14],[19,31]}$. Note how each set for the former function is base for construction of set for the latter function.

Case 6: Let A_6 be the subclass of $A \setminus (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)$ containing functions $f^n_{[0,b],[c,2^n-1]}$ with following properties:

1. $b + 1 \geq 2^n - c$
2. $01^{\{n-1\}}$ is a falsepoint
3. $101^{\{n-2\}}$ is a truepoint
4. $b^{[3,n]} + 1 = c^{[3,n]}$

There is an equality now in condition number 4. That means that there is no more such a falsepoint $01x$ as used in case 5. But there are not truepoints $01x$ and $10x$ as well.

Lemma 3.7. *(Case A6) Let f^n be the function from class A_6. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be the minimum spannig sets of functions $f_1 = f^{n-2}_{[0,b-2^{n-2}]}$ and $f_2 = f^{n-2}_{[c-2^{n-1},2^{n-2}-1]}$ respectively. Now define the following set:*

$$\mathscr{T} = \{00\phi^{\{n-2\}}\} \cup \{\phi 1T | T \in \mathscr{T}_1\} \cup \{1\phi T | T \in \mathscr{T}_2\} \tag{3.9}$$

Then $\mathscr{T}$ is a minimum spanning set of f^n. Moreover there is an orthogonal set for f^n which has an outer point.

Proof. We show that $\mathscr{T}$ is a spanning set. There cannot be any falsepoint with leading 01 spanned by some vector. That would mean that there is a vector in $\mathscr{T}_1$ spanning same falsepoint of f_1. For similar reason there cannot be any falsepoint with leading 10 spanned by some vector. Thus there is no falsepoint spanned. Truepoints with trailing 00 are spanned by vector $00\phi^{\{n-2\}}$. Truepoints with either trailing 01 or 10 are spanned by appropriate vectors originated in $\mathscr{T}_1$ or $\mathscr{T}_2$. Let $11x$ be a truepoint. From assumptions of the lemma we know that $b^{[3,n]} + 1 = c^{[3,n]}$ holds. If $x \leq b^{[3,n]}$ than $11x$ is spanned by vector $\phi 1T$ for some $T \in \mathscr{T}_1$. If $b^{[3,n]} < x$ then $11x$ is spanned by vector $1\phi T$ for some $T \in \mathscr{T}_2$.

To prove minimality of $\mathscr{T}$ we now construct an orthogonal set $\mathscr{E}$ for f^n where $|\mathscr{T}| = |\mathscr{E}|$. Let $E'_1, \ldots, E'_{k_1}$ be the orthogonal set constructed by lemma 2.1 for $f^{n-2}_{[0,b']}$ where $b' = b^{[3,n]}$. Define now vectors $E_i = 01E'_i$ for $1 \leq i \leq k_1$. Similarly let $Y'_1, \ldots, Y'_{k_2}$ be the orthogonal set constructed by lemma 2.2 for $f^{n-2}_{[c',2^{n-2}-1]}$ where $c' = c^{[3,n]}$. And define vectors $Y_i = 10Y'_i$ for $1 \leq i \leq k_2$. Now we need to choose a truepoint from among the truepoints E_i or Y_j. We can chooce whichever truepoint and for each different choice we get a different orthogonal set as a result (see figure 3.6). Without loss of generality we choose E_1. Let define the set:

$$\mathscr{E} = \{00E_1^{[3,n]}, 11E_1^{[3,n]}\} \cup \{E_2, \dots, E_{k_1}\} \cup \{Y_1, \dots, Y_{k_2}\} \tag{3.10}$$

Note that we have leaved out vector E_1. It is evident that all vectors in $\mathscr{E}$ are trueponts. Only orthogonality for all pairs $T_1, T_2 \in \mathscr{E}$ remained to be shown in order to complete the proof. Again, we distinguish five cases:

1. $T_1 = 00E_1^{[3,n]} \wedge T_2 \in \{E_2, \dots, E_{k_1}\}$
2. $T_1 = 00E_1^{[3,n]} \wedge T_2 \in \{Y_1, \dots, Y_{k_2}\}$
3. $T_1 = 00E_1^{[3,n]} \wedge T_2 = 11E_1^{[3,n]}$
4. $T_1, T_2 \in \{E_2, \dots, E_{k_1}\}$
5. $T_1 \in \{E_2, \dots, E_{k_1}\} \wedge T_2 \in \{Y_1, \dots, Y_{k_2}\}$

The remaining cases are symmetric. Now let T be a ternary vector spanning both T_1 and T_2. We are going to show that T has to span some falsepoint as well.

Case 1: Since $T_2^{[1,2]} = E_1^{[1,2]}$ we see that vector T spans truepoint E_1 as well. By the construction of vectors E_1 and $T_2 = E_i$ by lemma 2.1 we know that T also spans a falsepoint $b+1$.

Case 2: We have $T_1^{[3,n]} = E_1^{[3,n]}$ and $T_2^{[1,2]} = 10$. So also number $10E_1^{[3,n]}$ is spanned. The number is falsepoint because it holds that $10^{\{n-1\}} \leq 10E_1^{[3,n]} \leq 10b^{[3,n]} = c-1$. Numbers on both sides are falsepoints.

Case 3: In this case number $10E_1^{[3,n]}$ is spanned by T. It is shown to be a falsepoint in case 2.

Case 4: Both vectors come from lemma 2.1. In its proof it's shown that each vector spanning both of them spans also vector $b+1$ which is a falsepoint. And so does T.

Case 5: We have $T_1^{[1,2]} = 01$ and $T_2^{[1,2]} = 10$ which implies that $T^{[1,2]} = \phi\phi$. Thus T spans number $10E_1^{[3,n]}$. Once again it is shown to be a falsepoint in case 2.

We just have proved that $\mathscr{E}$ is an orthogonal set for f^n. We constructed spanning set $\mathscr{T}_1$ and an orthogonal set $E_1, \dots, E_{k_1}$ by lemma 2.1. It also claims that $|\mathscr{T}_1| = k_1$. By the same argument we see that also the size of $\mathscr{T}_2$ is the same as the number of vectors Y_i. Therefore size of $\mathscr{T}$ is the same as the size of $\mathscr{E}$ and thus by lemma 1.2 the constructed spanning set is of minimum size. We complete the proof by the fact that $00E_1^{[3,n]}$ is an outer point of an orthogonal set $\mathscr{E}$ for f^n.

$$|\mathscr{T}| = |\mathscr{T}_1| + |\mathscr{T}_2| + 1 \tag{3.11}$$

$$|\mathscr{E}| = (k_1 - 1) + k_2 + 2 \tag{3.12}$$

□

Case 7: Let A_7 be the class $A \setminus (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6)$ of all remaining functions containing functions $f^n_{[0,b],[c,2^n-1]}$ with following properties:

1. $b + 1 \geq 2^n - c$
2. $01^{\{n-1\}}$ is a falsepoint
3. $101^{\{n-2\}}$ is a truepoint
4. $b^{[3,n]} + 1 > c^{[3,n]}$

Lemma 3.8. *(Case A7) Let f^n be the function from class A_7. Let $\mathscr{T}'$ be the minimal spanning set of function $f^{n-1}_{[0,b'],[c',2^{n-1}-1]}$ where $b' = b - 2^{n-2}$ and $c' = c - 2^{n-2}$. Then the following set is a spanning set of f^n:*

$$
\begin{aligned}
\mathscr{T} = &\{00\phi^{\{n-2\}}\} && (3.13)\\
&\cup \{\phi 1 T^{[2,n]} | T \in \mathscr{T}' \wedge T^{[1]} = 0\} && (3.14)\\
&\cup \{1\phi T^{[2,n]} | T \in \mathscr{T}' \wedge T^{[1]} = 1\} && (3.15)\\
&\cup \{\phi\phi T^{[2,n]} | T \in \mathscr{T}' \wedge T^{[1]} = \phi\} && (3.16)
\end{aligned}
$$

Moreover there is an orthogonal set for f^n which has an outer point.

Proof. We want to show that $\mathscr{T}$ is a spanning set. Note that there is a one to one correspondence between truepoints of $f^{n-1}_{[0,b'],[c',2^{n-1}-1]}$ and truepoints of f^n from the interval $[2^{n-2}, 2^n - 2^{n-2} - 1]$. The same holds for falsepoints. Since all falsepoints of f^n lies in this subinterval we know that there cannot be any falsepoint of f^n spanned by $\mathscr{T}$ because that would imply that $\mathscr{T}'$ spans some falsepoint as well.

By the same argument all truepoints from the interval $[2^{n-2}, 2^n - 2^{n-2} - 1]$ are spanned by $\mathscr{T}$. All truepoints with 00 as their MSBs are spanned by vector $00\phi^{\{n-2\}}$. At last let $11x$ be the truepoint of f^n. If $x \leq b^{[3,n]}$ then there is a vector in $\mathscr{T}'$ spanning number $0x$. On the other hand if $x \geq b^{[3,n]} + 1 > c^{[3,n]}$ then there is a vector in $\mathscr{T}'$ spanning number $1x$. In both cases vector $T \in \mathscr{T}$ furnished from from vector spanning $0x$ or $1x$ spans truepoint $11x$ as well. ☐

Case A7 uses a recursion to solve the problem. The recursive call raised by this case cannot fall into cases A1, A2 or A3. We see the possible transitions between cases in figure 3.2. In lemma 3.8 we have proved just correctness of the spanning set. The proof of minimality of it is split into four cases. They are distinquished by the case which applies on the function in a recursive call raised by A7.

Observation 3.1. *In proofs of lemmas $A_1, \ldots, A_6$ minimality is proven by explicitly constructing orthogonal sets with the same size as spanning set.*

Lemma 3.9. *(A7.4) Let function f^n belong to subclass A_7, $b' = b - 2^{n-2}$ and $c' = c - 2^{n-2}$. Now denote by f^{n-1}_{rec} function $f^{n-1}_{[0,b'],[c',2^{n-1}-1]}$. Let f^{n-1}_{rec} belong to subclass A_4. Than the spanning set of f^n generated by lemma 3.8 is of minimum cardinality. Moreover there exists an orthogonal set for f^n which contains an outer point.*

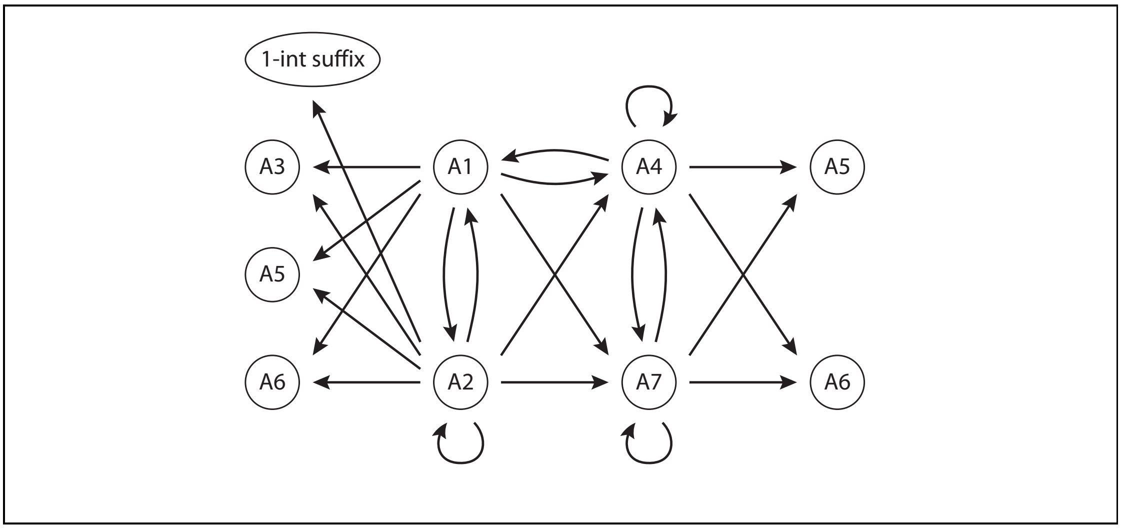

Figure 3.7: Graph of possible transitions between cases in solution to functions of type A

Proof. Note that function f_{rec}^{n-1} is a function which recursive solution is used to construct a spanning set in case 7. It is what we got if we reduce the number of variables by one and as a function values take function values of f^n in subinterval $[2^{n-2}, 2^{n-1} + 2^{n-2} - 1]$. This is equivalent to deminishing all numbers by 2^{n-2}. From the fact that f_{rec}^{n-1} belongs to case 4 and by the observation 3.1 we see that maximum orthogonal set for f_{rec}^{n-1} is of the same size as minimum spanning set for f_{rec}^{n-1}. Size of minimum spanning set for f^n furnished by case 7 is one more than the size of minimum spanning set for f_{rec}^{n-1}. Thus to prove its minimality it is enough to find an orthogonal set for f^n which size is one more than the size of maximum orthogonal set for f_{rec}^{n-1}.

We take the orthogonal set for f_{rec}^{n-1} generated by lemma 3.5 and shift it onto original place according to f^n by adding 2^{n-2} to all its members. With small further modifications we end up with an orthogonal set for f^n with one more vector. Let $E_1, \dots, E_j$ be a maximum orthogonal set for f_{rec}^{n-1} produced by lemma 3.5. From its proof we now that it contains a truepoint $001^{\{n-3\}}$. Without loss of generality we assume that $E_1 = 001^{\{n-3\}}$. We construct an orthogonal set $\mathscr{E}$ for f^n as follows:

$$\mathscr{E}_0 = \{01E^{[2,n-1]} | 2 \leq i \leq j \wedge E_i^{[1]} = 0\} \tag{3.17}$$

$$\mathscr{E}_1 = \{10E^{[2,n-1]} | 2 \leq i \leq j \wedge E_i^{[1]} = 1\} \tag{3.18}$$

$$\mathscr{E} = \mathscr{E}_0 \cup \mathscr{E}_1 \cup \{0001^{\{n-3\}}, 1101^{\{n-3\}}\} \tag{3.19}$$

Note that we have leaved out vector E_1 and added two others. The size of $\mathscr{E}$ is one more than the size of maximum orthogonal set for f_{rec}^{n-1}. We just need to show it to be orthogonal. All numbers in $\mathscr{E}$ are truepoints. We need to show them to be pairwise orthogonal. Let $T_1, T_2 \in \mathscr{E}$ be truepoints and T ternary vector spanning them.

Case (a): $T_1 = 0001^{\{n-3\}}$ (resp. $T_1 = 1101^{\{n-3\}}$) and $T_2 \in \mathscr{E}_0$. Since $T_2^{[1,2]} = 01$ vector T has to span truepoint $0101^{\{n-3\}} = 01E_1^{[2,n-1]}$ as well. And we know from lemma 3.5 that $01E_1^{[2,n-1]}$ and $T_2 = 01E_i^{[2,n-1]}$ (for some $i \neq 1$) are orthogonal given f^n.

Case (b): $T_1 = 0001^{\{n-3\}}$ (resp. $T_1 = 1101^{\{n-3\}}$) and $T_2 \in \mathscr{E}_1$. Since $T_2^{[1,2]} = 10$. Vector T spans a falsepoint 1001^{n-3} as well.

Case (c): $T_1 = 0001^{\{n-3\}}$ and $T_2 = 1101^{\{n-3\}}$. Again vector T spans a falsepoint 1001^{n-3}.

Case (d): Both T_1 and T_2 have either 10 or 01 as their MSBs. Then both truepoints belongs to the original orthogonal set for f_{rec}^{n-1} furnished by lemma 3.5. Therefore we know them to be orthogonal.

At last, note that $0001^{\{n-3\}}$ is an outer point of set $\mathscr{E}$. ☐

Lemma 3.10. *(A7.5) Let function f^n belong to subclass A_7, $b' = b - 2^{n-2}$ and $c' = c - 2^{n-2}$. Now denote by f_{rec}^{n-1} function $f_{[0,b'],[c',2^{n-1}-1]}^{n-1}$. Let f_{rec}^{n-1} belong to subclass A_5. Than the spanning set of f^n generated by lemma 3.8 is of minimum cardinality. Moreover there is an orthogonal set for f^n which contains an outer point.*

Proof. Again function f_{rec}^{n-1} is a function which recursive solution is used to construct a spanning set in case 7. It is the same function as defined in lemma 3.9. From the fact that f_{rec}^{n-1} belongs now to case 5 and by the observation 3.1 we see that maximum orthogonal set for f_{rec}^{n-1} is of the same size as minimum spanning set for f_{rec}^{n-1}. Size of minimum spanning set for f^n furnished by case 7 is one more than the size of minimum spanning set for f_{rec}^{n-1}. Thus to prove its minimality it is enough to find an orthogonal set for f^n which size is one more than the size of maximum orthogonal set for f_{rec}^{n-1}.

We take the orthogonal set for f_{rec}^{n-1} generated by lemma 3.6 and shift it onto original place according to f^n by adding 2^{n-2} to all its members. After few modifications we end up with an orthogonal set for f^n with one more vector. Let $E_1, \ldots, E_j$ be a maximum orthogonal set for f_{rec}^{n-1} produced by lemma 3.6. From its proof we know that it contains a truepoint $00x$ of length $n-1$ such that $01x$ and $10x$ are falsepoints of f_{rec}^{n-1}. Without loss of generality we assume that $E_1 = 00x$. We construct an orthogonal set $\mathscr{E}$ for f^n as follows:

$$\mathscr{E}_0 = \{01E^{[2,n-1]} | 2 \leq i \leq j \wedge E_i^{[1]} = 0\} \tag{3.20}$$

$$\mathscr{E}_1 = \{10E^{[2,n-1]} | 2 \leq i \leq j \wedge E_i^{[1]} = 1\} \tag{3.21}$$

$$\mathscr{E} = \mathscr{E}_0 \cup \mathscr{E}_1 \cup \{000x, 110x\} \tag{3.22}$$

Note that we have leaved out vector E_1 and added two others. The size of $\mathscr{E}$ is one more than the size of maximum orthogonal set for f_{rec}^{n-1}. We just need to show it to be orthogonal. All numbers in $\mathscr{E}$ are truepoints. We need to show them to be pairwise orthogonal. Let $T_1, T_2 \in \mathscr{E}$ be truepoints and T ternary vector spanning them.

Case (a): $T_1 = 000x$ (resp. $T_1 = 110x$) and $T_2 \in \mathscr{E}_0$. We have that $T_2^{[1,2]} = 01$. Vector T has to span truepoint $010x = 01E_1^{[2,n-1]}$ as well. And we know from lemma 3.6 that $01E_1^{[2,n-1]}$ and $T_2 = 01E_i^{[2,n-1]}$ (for some $i \neq 1$) are orthogonal given f^n.

Case (b): $T_1 = 000x$ (resp. $T_1 = 110x$) and $T_2 \in \mathscr{E}_1$. We have that $T_2^{[1,2]} = 10$. Vector T spans a falsepoint $100x$ as well.

Case (c): $T_1 = 000x$ and $T_2 = 110x$. Again vector T spans a falsepoint $100x$.

computeDNF-typeA(b,c,n)

Input: Numbers b, c and n such that $0 \leq b < c < 2^n$
Output: Spanning set of function $f^n_{[0,b],[c,2^n-1]}$ of minimum cardinality

if $b+1=c$ **then return** $\{\phi^{\{n\}}\}$

if $b+1 < 2^n - c$ **then return** *computeDNF-typeA*($\bar{c}$, $\bar{b}$, n)

if $b = 2^{n-1}-1$ **then return** $\{0\phi^{n-1}\} \cup$ *suffix*(c,2^n-1,n)

if $b > 2^{n-1}-1$ **then**
  $recT :=$ *computeDNF-typeA*($b-2^{n-1}$, $c-2^{n-1}$, $n-1$)
  return $\{0\phi^{n-1}\} \cup \{1T | T \in recT\}$

if $b^{[2,n]} < c^{[2,n]}$ **then return** *prefix*(0,b,n) $\cup$ *suffix*(c,2^n-1,n)

if $c \geq 2^{n-1}+2^{n-2}$ **then**
  $recT :=$ *computeDNF-typeA*($b-2^{n-2}$, $c-2^{n-1}$, $n-1$)
  return $\{00\phi^{n-2}\} \cup \{T^{[1]}1T^{[2,n]} | T \in recT\}$

if $b^{[3,n]}+1 < c^{[3,n]}$ **then return** *prefix*(0,b,n) $\cup$ *suffix*(c,2^n-1,n)

if $b^{[3,n]}+1 = c^{[3,n]}$ **then**
  $P :=$ *prefix*($0, b-2^{n-2}, n-2$)
  $S :=$ *suffix*($c-2^{n-1}, 2^{n-2}-1, n-2$)
  return $\{00\phi^{n-2}\} \cup \{\phi 1T | T \in P\} \cup \{1\phi T | T \in S\}$

if $b^{[3,n]}+1 > c^{[3,n]}$ **then**
  $recT :=$ *computeDNF-typeA*($b-2^{n-2}$, $c-2^{n-2}$, $n-1$)
  $recT_0 := \{\phi 1T^{[2,n]} | T \in recT \wedge T^{[1]} = 0\}$
  $recT_1 := \{1\phi T^{[2,n]} | T \in recT \wedge T^{[1]} = 1\}$
  $recT_\phi := \{\phi T | T \in recT \wedge T^{[1]} = \phi\}$
  return $\{00\phi^{n-2}\} \cup recT_0 \cup recT_1 \cup recT_\phi$

Figure 3.8: Construction of minimum spanning set for 2-interval function of type A

Case (d): Both T_1 and T_2 have either 10 or 01 as their MSBs. Then both truepoints belongs to the original orthogonal set for f^{n-1}_{rec} furnished by lemma 3.6. Therefore we know them to be orthogonal.

To the end, note that truepoint $000x$ is an outer point of set $\mathscr{E}$. $\square$

Lemma 3.11. *(A7.6) Let function f^n belong to subclass A_7, $b' = b - 2^{n-2}$ and $c' = c - 2^{n-2}$. Now denote by f^{n-1}_{rec} function $f^{n-1}_{[0,b'],[c',2^{n-1}-1]}$. Let f^{n-1}_{rec} belong to subclass A_6. Than the spanning set of f^n generated by lemma 3.8 is of minimum cardinality. Moreover there is an orthogonal set for f^n which contains an outer point.*

Proof. Again function f_{rec}^{n-1} is a function which recursive solution is used to construct a spanning set in case 7. It is the same function as defined in lemma 3.9. By the same argument as in lemma 3.9 it is enough to find an orthogonal set for f^n which size is one more than the size of maximum orthogonal set for f_{rec}^{n-1}.

We take the orthogonal set for f_{rec}^{n-1} generated by lemma 3.7 and shift it onto original place according to f^n by adding 2^{n-2} to all its members. After modifying it we end up with an orthogonal set for f^n with one more vector. Let $E_1, \dots, E_j$ be a maximum orthogonal set for f_{rec}^{n-1} produced by lemma 3.7. From its proof we now that it contains an outer point $00x$ of length $n-1$. Without loss of generality we assume that $E_1 = 00x$. We construct an orthogonal set $\mathscr{E}$ for f^n as follows:

$$\mathscr{E}_0 = \{01E^{[2,n-1]} | 2 \le i \le j \wedge E_i^{[1]} = 0\} \tag{3.23}$$
$$\mathscr{E}_1 = \{10E^{[2,n-1]} | 2 \le i \le j \wedge E_i^{[1]} = 1\} \tag{3.24}$$
$$\mathscr{E} = \mathscr{E}_0 \cup \mathscr{E}_1 \cup \{000x, 110x\} \tag{3.25}$$

Once again we have constructed a set of truepoints of f^n of size one more than the size of maximum orthogonal set for f_{rec}^{n-1}. We show an orthogonality. Let $T_1, T_2 \in \mathscr{E}$ be truepoints and T ternary vector spanning them.

Case (a): $T_1 = 000x$ (resp. $T_1 = 110x$) and $T_2 \in \mathscr{E}_0$. We have that $T_2^{[1,2]} = 01$. Vector T has to span truepoint $010x = 01E_1^{[2,n-1]}$ as well. And we know from the proof of lemma 3.7 that $01E_1^{[2,n-1]}$ and $T_2 = 01E_i^{[2,n-1]}$ (for some $i \neq 1$) are orthogonal given f^n.

Case (b): $T_1 = 000x$ (resp. $T_1 = 110x$) and $T_2 \in \mathscr{E}_1$. We have that $T_2^{[1,2]} = 10$ and so T spans number $100x$ as well. We are going to show that it is a falsepoint of f^n. From the proof of lemma 3.7 we know the form of orthogonal set $E_1, \dots, E_j$. More specifically we know that there are truepoints $00x$ and $11x$ among them. These truepoints are spanned by ternary vector $\phi\phi x$. Since those two points are members of an orthogonal set there has to be some falsepoint spanned by $\phi\phi x$. Only two possible candidates are $01x$ and $10x$. But we know that $00x$ is an outer point so it follows that also $01x$ is a truepoint for f_{rec}^{n-1}. Therefore number $10x$ is a falsepoint of f_{rec}^{n-1} which also implies that $100x$ is a falsepoint of f^n.

Case (c): $T_1 = 000x$ and $T_2 = 110x$. Again vector T spans a falsepoint $100x$.

Case (d): Both T_1 and T_2 have either 10 or 01 as their MSBs. Then both truepoints belongs to the original orthogonal set for f_{rec}^{n-1} furnished by lemma 3.6. Therefore we know them to be orthogonal.

At last, note that truepoint $000x$ is an outer point of set $\mathscr{E}$. ☐

Lemma 3.12. *(A7.7) Let function f^n belong to subclass A_7, $b' = b - 2^{n-2}$ and $c' = c - 2^{n-2}$. Now denote by f_{rec}^{n-1} function $f_{[0,b'],[c',2^{n-1}-1]}^{n-1}$. Let f_{rec}^{n-1} belong to subclass A_7. Than the spanning set of f^n generated by lemma 3.8 is of minimum cardinality. Moreover there is an orthogonal set for f^n which contains an outer point.*

Proof. Again function f_{rec}^{n-1} is a function which recursive solution is used to construct a spanning set in case 7. It is the same function as defined in lemma 3.9. By the same argument as in lemma 3.9 it is enough to find an orthogonal set for f^n which size is one more than the size of maximum orthogonal set for f_{rec}^{n-1}.

We take the orthogonal set for f_{rec}^{n-1} generated by lemma 3.8 and shift it onto original place according to f^n by adding 2^{n-2} to all its members. After modifying it we end up with an orthogonal set for f^n with one more vector. Let $E_1, \ldots, E_j$ be a maximum orthogonal set for f_{rec}^{n-1} produced by one of lemmas 3.9, 3.10, 3.11, 3.12. From its proof we now that it contains an outer point $00x$ of length $n-1$. Without loss of generality we assume that $E_1 = 00x$. We construct an orthogonal set $\mathscr{E}$ for f^n as follows:

$$\mathscr{E}_0 = \{01E^{[2,n-1]} | 2 \leq i \leq j \wedge E_i^{[1]} = 0\} \tag{3.26}$$
$$\mathscr{E}_1 = \{10E^{[2,n-1]} | 2 \leq i \leq j \wedge E_i^{[1]} = 1\} \tag{3.27}$$
$$\mathscr{E} = \mathscr{E}_0 \cup \mathscr{E}_1 \cup \{000x, 110x\} \tag{3.28}$$

Once again we have constructed a set of truepoints of f^n of size one more than the size of maximum orthogonal set for f_{rec}^{n-1}. We show an orthogonality. Let $T_1, T_2 \in \mathscr{E}$ be truepoints and T ternary vector spanning them.

Case (a): $T_1 = 000x$ (resp. $T_1 = 110x$) and $T_2 \in \mathscr{E}_0$. We have that $T_2^{[1,2]} = 01$. Vector T has to span truepoint $010x = 01E_1^{[2,n-1]}$ as well. And we know from the proof of lemma 3.8 that $01E_1^{[2,n-1]}$ and $T_2 = 01E_i^{[2,n-1]}$ (for some $i \neq 1$) are orthogonal given f^n.

Case (b): $T_1 = 000x$ (resp. $T_1 = 110x$) and $T_2 \in \mathscr{E}_1$. We have that $T_2^{[1,2]} = 10$ and so T spans number $100x$ as well. We show that it is a falsepoint of f^n. From the proof of lemma 3.8 we know that there are truepoints $00x$ and $11x$ among vectors $E_1, \ldots, E_j$. These truepoints are spanned by ternary vector $\phi\phi x$. Since those two points are members of an orthogonal set there has to be some falsepoint spanned by $\phi\phi x$. Only two possible candidates are $01x$ and $10x$. But we know that $00x$ is an outer point so it follows that also $01x$ is a truepoint for f_{rec}^{n-1}. Therefore number $10x$ is a falsepoint of f_{rec}^{n-1} which also implies that $100x$ is a falsepoint of f^n.

Case (c): $T_1 = 000x$ and $T_2 = 110x$. Again vector T spans a falsepoint $100x$.

Case (d): Both T_1 and T_2 have either 10 or 01 as their MSBs. Then both truepoints belongs to the original orthogonal set for f_{rec}^{n-1} furnished by lemma 3.6. Therefore we know them to be orthogonal.

At last, note that truepoint $000x$ is an outer point of set $\mathscr{E}$. ☐

Algorithm on figure 3.8 constructs a minimum spanning set for 2-interval function of type A. Resulting set is in general not the only one with minimum size. For example let f^n be function with only two falsepoints namely $01^{\{n-1\}}$ and $10^{\{n-1\}}$. On figure 3.9 we can see how the number of spanning sets for this function grows with number of variables. Even though not all spanning sets are considered in those counts (see section 5.2 for detailed information.) they grow as fast as $n!$ function. Data on figure 3.9 were found by algorithm on figure 5.4.

n	2	3	4	5	6	7	...
#spanSets	1	2	6	24	120	720	...

Figure 3.9: Number of spanning sets of function $f^n_{[0,2^{n-1}-2],[2^{n-1}+1,2^n-1]}$ formed by maximal vectors only

Recall that algorithm on figure 3.8 can be easily modified to produce not just minimum spanning set of a function but an orthogonal set of the same size as well. That can be seen from the proofs of involved lemmas. Thus we have following observation.

Observation 3.2. *Hypothesis 1.1 holds for the class of* 2*-interval functions of type A.*

Now that we have the optimization solution for functions of type A we are able to assess an error of approximation algorithm on figure 3.1. In lemma 3.1 we have shown that $|\mathscr{T}_{apr}| < 2|\mathscr{T}_{opt}|$. Now we can show that this bound for error is tight.

Lemma 3.13. *Let $f^n_{[0,b],[c,2^n-1]}$ be a function and $\mathscr{T}$ be its minimum spanning set. Moreover let $\mathscr{T}_1$ and $\mathscr{T}_2$ be minimal spanning sets of $f^n_{[0,b]}$ and $f^n_{[c,2^n-1]}$ respectively. Then $|\mathscr{T}_1| \cup |\mathscr{T}_2| < 2|\mathscr{T}|$ and this is a tight bound for an error.*

Proof. We know from the lemma 3.1 that $|\mathscr{T}_1| \cup |\mathscr{T}_2| < 2|\mathscr{T}|$. Only we need to prove is that this inequality does not hold for any constant less then 2. We do that by constructing a sequence of functions such that sequence of their errors approaches 2 in a limit. Therefore there shall be a counterexample to $|\mathscr{T}_1| \cup |\mathscr{T}_2| < c|\mathscr{T}|$ for any $c < 2$ among those functions.

Define function $f^n_{[0,b],[c,2^n-1]}$ to be a function where $b = 2^{n-1}-2$ and $c = 2^{n-1}+1$. We claim that sequence of such functions $\{f^n\}_{n\geq 2}$ has the desired property. We define $approx(f^n)$ as size of spanning set produced by approximation algorithm. By lemmas 3.1 and 2.1 it is the case that $approx(f^n)$ equals to the number of 1-bits in number $b+1$ plus the number of 0-bits in number $c-1$. The number of 1-bits in $b+1 = 2^{n-1}-1$ is $n-1$. The number of 0-bits in $c-1 = 2^{n-1}$ is $n-1$ as well. So $approx(f^n) = 2(n-1)$. Similarly we define $opt(f^n)$ as size of spanning set produced by optimisation algorithm. Note that function f^3 is solved by case A7 in optimisation algorithm and that function to be solved recursively would be f^2. This holds in general. Function f^i (for $i \geq 3$) is solved by case A7 and function to be solved recursively is f^{i-1}. Recall that the size of minimum spanning set of f^i is one plus the size of minimum spanning set of f^{i-1}. Since we know that $opt(f^2) = 2$ we have that $opt(f^n) = n$. A limit of sequence of approximation errors of functions $\{f^n\}_{n\geq 2}$ is

$$\lim_{n\to\infty} \frac{approx(f^n)}{opt(f^n)} = \lim_{n\to\infty} \frac{2(n-1)}{n} = 2 \tag{3.29}$$

□

4. Spanning sets of general 2-interval functions

We have constructed an optimization algorithm for functions of type A. In this chapter we start by presenting a counterexamples to the hypothesis 1.1 from the class of 2-interval functions of type B and C. They were found using programs described in chapter 5. There is no counterexample among functions of less then four variables.

We start by counterexample from functions of type B. It is function $f^4_{[0,4],[9,14]}$. The size of its maximum orthogonal set is 4 whereas the size of its minimum spanning set is 5. All eight possible orthogonal sets of maximum size of the function can be seen on figure 4.1. The number of different spanning sets for the function found is 21. Those are not all possible spanning sets. See chapter 5.2 for detailed information. Dump out of all spanning sets would be waste of space. We present at least one of them as an example. Note that sum of sizes of spanning sets for $f^4_{[0,4]}$ and $f^4_{[9,14]}$ are 5. So this "approximate" solution is in this case an optimal as well. We can see this solution on figure 4.2. Note that also intuitively there is no way how to rearrange ternary vectors such that it would be possible to omit one of them.

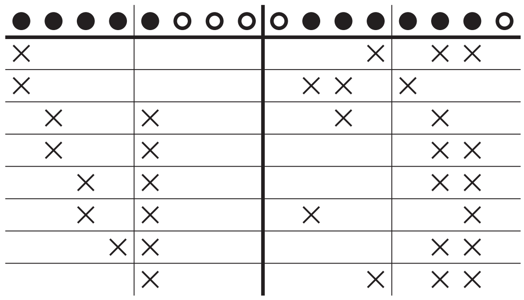

Figure 4.1: All orthogonal sets of maximum size of function $f^4_{[0,4],[9,14]}$

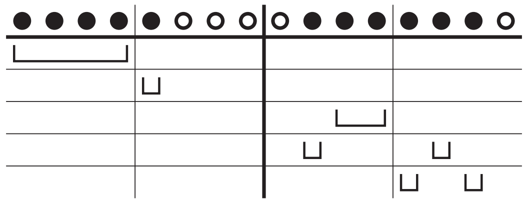

Figure 4.2: Example of spanning set of minimum size of function $f^4_{[0,4],[9,14]}$

Counterexample from functions o type C is function $f^4_{[1,10],[12,12]}$. The size of its maximum orthogonal set is 4 and size of its minimum spanning set is 5. All possible orthogonal sets of size 4 and an example of spanning set of size 5 can be seen on figures 4.3 and 4.4.

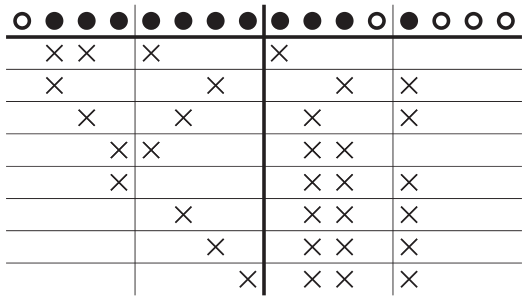

Figure 4.3: All orthogonal sets of maximum size of function $f^4_{[1,10],[12,12]}$

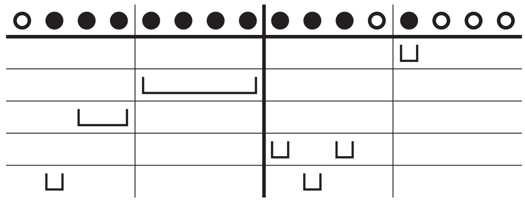

Figure 4.4: Example of spanning set of minimum size of function $f^4_{[1,10],[12,12]}$

Observation 4.1. *Hypothesis 1.1 does not hold for 2-interval functions of type B and of type C. Thus it also does not hold for 2-interval functions in general.*

Necessary part of construction of algorithm is formal proof of its correctness. Proofs for algorithms for 1-interval functions and for 2-interval functions of type A are based on validity of hypothesis 1.1. Therefore, by rejecting it we have lost the only tool we had available for proving minimality of constructed spanning sets for other classes. We can try to establish an approximation algorithm for general 2-interval functions. Once again we can solve each interval separately by optimal algorithm for 1-interval function.

approxDNF(a,b,c,d,n)

Input: Numbers a, b, c, d and n such that $0 \leq a \leq b < c \leq d < 2^n$
Output: Spanning set of function $f^n_{[a,b],[c,d]}$

$\mathscr{T}_1 :=$ *computeDNF*(a, b, n)
$\mathscr{T}_2 :=$ *computeDNF*(c, d, n)
return $\mathscr{T}_1 \cup \mathscr{T}_2$

Figure 4.5: Approximation of minimum spanning set for 2-interval function

It is clear that returned solution is correct spanning set. We want to examine how much this solution differs from an optimal one.

Lemma 4.1. *Let f^n be a 2-interval function from class B or from class C and let $\mathscr{T}_{opt}$ be its spanning set of minimum size. Moreover let $\mathscr{T}_{approx}$ be a spanning set of f^n returned by approximation algorithm in figure 4.5. Then it holds that*

$$\frac{|\mathscr{T}_{approx}|}{|\mathscr{T}_{opt}|} \geq 2 \tag{4.1}$$

Proof. Let a be n-bit number. Define 2-interval function as $f^{n+1}_{[0a,0a],[1a,1a]}$. It is an 2-interval function and we can see that minimum spanning set size is 1 as $\{0a\}$ is an orthogonal set and $\{\phi a\}$ is a spanning set both of size 1. However, spanning set returned by approximation algorithm is $\{0a, 1a\}$. Therefore an error of approximation algorithm is at least 2. If $a = 0 \vee a = 2^n - 1$ then $f^{n+1}_{[0a,0a],[1a,1a]}$ is a function of type B. Otherwise it is function of type C. Thus this lower bound for error holds for both classes. □

Now we want to proove that 2 is also an upper bound. In proof of this for functions of type A (lemma 3.1) we have taken an orthogonal set constructed by lemma 2.1 or 2.2 for one of two intervals. Namely that which requires more vectors to be spanned. We then proved that the set is also an orthogonal set for whole 2-interval function. That implied that minimum spanning set size for one of intervals is a lower bound for size of minimum spanning set for whole 2-interval function. But we cannot do that so directly in case of functions of type C. We are going to explain why it is so.

Let $f^n_{[a,b],[c,d]}$ be a 2-interval function and let S be an orthogonal set for $f^n_{[a,b]}$ which is furnished by the proof of theorem 2.1. Then we would fail to prove that S is also an orthogonal set for $f^n_{[a,b],[c,d]}$. It is because it is not true in general. Construction of S in theorem 2.1 is based on some falsepoints of $f^n_{[a,b]}$. But those are not necessarily falsepoints of $f^n_{[a,b],[c,d]}$.

So we would like to adjust the process of construction of an orthogonal set for function $f^n_{[a,b]}$ (for some a and b) in such a way that the outcome would also be an orthogonal set for functions $f^n_{[a,b],[c,d]}$ and $f^n_{[a',b'],[a,b]}$. So we have to think of the way how to construct an orthogonal set for $f^n_{[a,b]}$ without relying on falsepoints which can be spanned by second interval of truepoints afterwards. As a result, only falsepoints which we can base our new method on are $a-1$ (if $a \neq 0$) and $b+1$ (if $b \neq 2^n - 1$). That leads us to the following definition.

Definition 4.1 (General spanning set). *For two n-bit numbers $a \leq b$ the set of ternary vectors $\mathscr{T}$ is a general spanning set of function $f^n_{[a,b]}$ if following conditions hold:*

1. *Every number in interval $[a, b]$ is spanned by some vector from $\mathscr{T}$.*
2. *If $a \neq 0$ then no vector from $\mathscr{T}$ spans number $a - 1$.*
3. *If $b \neq 2^n - 1$ then no vector from $\mathscr{T}$ spans number $b + 1$.*

We say that $\mathscr{T}$ generally spans function $f^n_{[a,b]}$.

Note that this definition requires f^n to be a 1-interval function while the definition 1.14 of spanning set was for all boolean functions. But applied on 1-interval functions spanning set of an interval as defined in 1.14 means that its vectors span numbers from the interval and do not span any number from outside of the interval. General spanning set needs only to span numbers from the interval itself and border it by not spanned numbers. It does not depend on behaviour of vectors on other numbers.

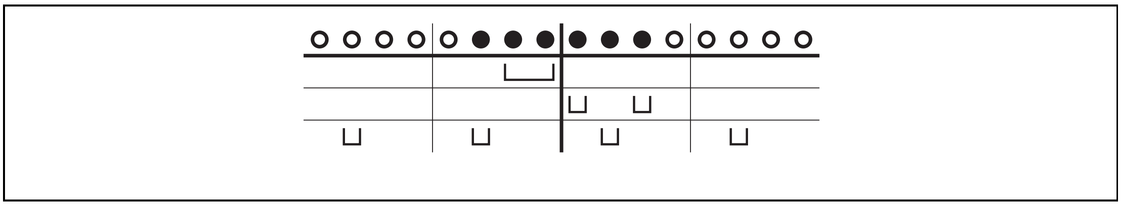

Figure 4.6: General spanning set of size 3 of function $f^4_{[5,10]}$

Observation 4.2. *Every spanning set of function $f^n_{[a,b]}$ is also general spanning set of the same function.*

At this point it is natural to ask if minimum spanning set of 1-interval function produced by optimisation algorithm presented in chapter 2 (figure 2.4) is a minimum general spanning set of the same 1-interval function as well. Counterexample to this hypothesis is function $f^4_{[5,10]}$. Minimum spanning set of this function has size 4 and example of it is $\{0101, 011\phi, 1010, 100\phi\}$. But there is a general spanning set of size 3 which looks like this $\{011\phi, 10\phi0, \phi\phi01\}$. It can be seen on figure 4.6.

Nevertheless this counterexample requires to have some numbers spanned on both sides of an interval which is generally spanned. But that never happens in case of adding one interval of truepoints to furnish a 2-interval function. That leads us to another modification of notion of spanning set.

Definition 4.2 (Left (right) general spanning set)**.** *Let $f^n_{[a,b]}$ be a function to be spanned and let $\mathscr{T}$ be the set of ternary vectors of length n which forms general spanning set of $f^n_{[a,b]}$. Then:*

1. *$\mathscr{T}$ is left general spanning set of $f^n_{[a,b]}$ if no vector from $\mathscr{T}$ spans any number from interval $[0, a-1]$ (which is empty when $a = 0$).*

2. *$\mathscr{T}$ is right general spanning set of $f^n_{[a,b]}$ if no vector from $\mathscr{T}$ spans any number from interval $[b+1, 2^n-1]$ (which is empty when $b = 2^n - 1$).*

We say that $\mathscr{T}$ generally spans $f^n_{[a,b]}$ from the left (from the right respectively).

Observation 4.3. *Every spanning set of function $f^n_{[a,b]}$ is also both left and right general spanning set of the same function.*

Definition 4.3 (Left-sided (right-sided) point of orthogonal set)**.** *Let $\mathscr{E}$ be an orthogonal set for function f^n. We say that $e \in \mathscr{E}$ is a left-sided point of $\mathscr{E}$ if $e^{[1]} = 0 \wedge 1e^{[2,n]}$ is a falsepoint of f^n. Similarly $e \in \mathscr{E}$ is a right-sided point of $\mathscr{E}$ if $e^{[1]} = 1 \wedge 0e^{[2,n]}$ is a falsepoint of f^n.*

In the proof of the following lemma we will use notions of an "orthogonal set" and of "right-sided point". Definition of both of them strongly depends on falsepoints of a function. In proofs so far we used the notion of falsepoint as a number which cannot be spanned by any ternary vector in spanning set. This is not true anymore for left (right, general) spanning sets. Therefore we can consider a falsepoint as not spanned by any spanning set only if the definition of spanning set guarantees that.

Definition 4.4 (Left (right) general orthogonal pair). *Let x and y be two truepoints of $f^n_{[a,b]}$. We say that they are left generally orthogonal for $f^n_{[a,b]}$ if each vector spanning both of them necessarily spans falsepoint $b+1$ or some falsepoint from the interval $[0, a-1]$. We say that they are right generally orthogonal for $f^n_{[a,b]}$ if each vector spanning both of them necessarily spans falsepoint $a-1$ or some falsepoint from the interval $[b+1, 2^n-1]$.*

Definition 4.5 (Left (right) general orthogonal set). *Let $\mathscr{E}$ be an orthogonal set for $f^n_{[a,b]}$. We say that $\mathscr{E}$ is left general orthogonal set for $f^n_{[a,b]}$ if all $x, y \in \mathscr{E}$ forms an left generally orthogonal pair for $f^n_{[a,b]}$. Similarly we say that $\mathscr{E}$ is right general orthogonal set for $f^n_{[a,b]}$ if all $x, y \in \mathscr{E}$ forms an right generally orthogonal pair for $f^n_{[a,b]}$. We can omit word "generally" if it does not cause a confusion.*

Lemma 4.2. *Orthogonal sets constructed for prefix (suffix) 1-interval functions by proof of lemma 2.1 (lemma 2.2) are also general orthogonal sets for the same function.*

Proof. After revision of proofs of lemmas we see that only falsepoints they rely on are $b+1$ in prefix case and $a-1$ in suffix case. Thus an orthogonal sets they produce are general orthogonal sets. ☐

Theorem 4.1. *Minimum spanning set of function $f^n_{[a,b]}$ constructed by optimisation algorithm for 1-interval function (figure 2.4) is minimum left (right) general spanning set of the same function.*

Proof. By observation 4.3 set constructed by optimisation algorithm for 1-interval function is left (right resp.) general spanning set of the same function. Thus it is enough to prove the minimality. Without loss of generality we are going to show it for case of left general spanning sets. The right case is proven similarly. We prove the lemma by showing how to explicitly construct an left general orthogonal set for $f^n_{[a,b]}$ which has the same size as a spanning set constructed by optimisation algorithm for 1-interval function. Be aware of important differance between spanning set and left general spanning set in this proof. The same holds for orthogonal sets.

Function which is identically true has spanning set $\{\phi^{\{n\}}\}$ which obviously fulfills the statement of the lemma. We will use lemma 2.1 and lemma 2.2 to deal with prefix and suffix cases. Orthogonal sets their proofs produce are left general orthogonal sets by lemma 4.2. So for the rest of proof we assume that $0 < a \leq b < 2^n - 1$. We construct the set using induction by number of variables exactly as in construction of a spanning set. Induction hypothesis we will be maintaining is the following: for each 1-interval function of less than n variables there is an left general orthogonal set of the same size as its spanning set moreover containing a right-sided point (as defined in 4.4).

Base case is for $n = 2$. There is only one such a function namely $f^2_{[1,2]}$. Its minimum spanning set has size 2. An left general orthogonal set of the same size is $\{01, 10\}$ where 10 is its right-sided point. If $a^{[1]} = b^{[1]}$ then we can just fix the first bit of a left general orthogonal set under construction. So we can also assume that $a^{[1]} = 0$ and $b^{[1]} = 1$. Now let induction hypothesis hold for $n-1$. We want to prove it for n. Let $\mathscr{T}$ be a minimum spanning set for function $f^n_{[a,b]}$ constructed by optimisation algorithm for 1-interval function. We distinquish four cases.

Case 1: We have $a^{[1,2]} = 01$ and $b^{[1,2]} = 10$. Recall that spanning set $\mathscr{T}$ is constructed as union of spanning set for $f^n_{[a,b'-1]}$ and $f^n_{[b',b]}$ where $b' = 2^{n-1}$. The former subfunction is solved by suffix case and the latter case by prefix case. Proofs of prefix and suffix cases constructs not only appropriate spanning sets $\mathscr{T}_1$ (for $f^n_{[a,b'-1]}$) and $\mathscr{T}_2$ (for $f^n_{[b',b]}$) but also an orthogonal sets $\mathscr{E}_1$ and $\mathscr{E}_2$ for treated functions. We have shown at the begining of this proof that they are also left general orthogonal sets. Moreover we know from proofs of prefix and suffix cases that $|\mathscr{E}_1| = |\mathscr{T}_1|$ and $|\mathscr{E}_2| = |\mathscr{T}_2|$. Also $\mathscr{E}_1$ and $\mathscr{E}_2$ are disjunct because intervals $[a, b'-1]$ and $[b', b]$ are disjunct. We claim that $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2$ is a left orthogonal set for f^n with desired properties. To show that it is a left orthogonal set for f^n let $e_1, e_2 \in \mathscr{E}$. If they are both from $\mathscr{E}_i$ then they are left orthogonal by left orthogonality of $\mathscr{E}_i$. If without loss of generality $e_1 \in \mathscr{E}_1$ and $e_2 \in \mathscr{E}_2$ then $e_1^{[1,2]} = 01$ and $e_2^{[1,2]} = 10$. Then any ternary vector spanning both of them has $\phi\phi$ as its MSBs. Such a ternary vector necessarily spans some falsepoint less then a. So it is a left orthogonal set for f^n with the same size as $\mathscr{T}$. Moreover any truepoint $e \in \mathscr{E}_2$ is a right-sided point of $\mathscr{E}$. It is because $e^{[1,2]} = 10$ and $00e^{[3,n]}$ is a falsepoint.

Case 2: We have $a^{[1,2]} = 00$ and $b^{[1,2]} = 10$. Both numbers have 0 as their second bit. Construct the function $f^{n-1}_{[a',b']}$ where $a' = a^{[1]}a^{[3,n]}$ and $b' = b^{[1]}b^{[3,n]}$. This function has $n-1$ variables so by induction hypothesis there is a left orthogonal set $\mathscr{E}_r$ for it of the same size as its minimum spanning set and containing right-sided point. Now construct the set like this:

$$\mathscr{E} = \{e^{[1]}0e^{[2,n-1]} | e \in \mathscr{E}_r\} \cup \{01(a-1)^{[3,n]}\} \tag{4.2}$$

We want to show that it is a left orthogonal set for $f^n_{[a,b]}$. Note that there is a one to one correspondence between numbers in interval $[a, 2^{n-2}-1]$ and numbers in interval $[a', 2^{n-2}-1]$. It is enough to remove 0-bit from second position to convert number from the former into the latter interval. The same holds for intervals $[2^{n-1}, b]$ and $[2^{n-2}, b']$. Thus all reasoning used to prove the left orthogonality of $\mathscr{E}_r$ for $f^{n-1}_{[a',b']}$ is also valid as a proof of the left orthogonality of $\mathscr{E} \setminus \{01(a-1)^{[3,n]}\}$ for $f^n_{[a,b]}$. Now let T be a ternary vector spanning both $01(a-1)^{[3,n]}$ and $e \in \mathscr{E} \setminus \{01(a-1)^{[3,n]}\}$. We know that $e^{[2]} = 0$ which implies that $T^{[2]} = \phi$. So T also spans falsepoint $a-1$. We have shown that $\mathscr{E}$ is a left orthogonal set for $f^n_{[a,b]}$. Recall that spanning set $\mathscr{T}$ constructed in this case has the size of minimum spanning set of $f^{n-1}_{[a',b']}$ plus one. This fact together with induction hypothesis and the way we have constructed $\mathscr{E}$ implies that $|\mathscr{E}| = |\mathscr{T}|$. Now let e_r be a right-sided point of $\mathscr{E}_r$. We know then that $e_r^{[1]} = 1$ and that $0e_r^{[2,n-1]}$ is a falsepoint of $f^{n-1}_{[a',b']}$. Thus $10e_r^{[2,n-1]} \in \mathscr{E}$ is a right-sided point of $\mathscr{E}$ since $00e_r^{[2,n-1]}$ is a falsepoint of $f^n_{[a,b]}$.

Case 3: We have $a^{[1,2]} = 01$ and $b^{[1,2]} = 11$. This case looks symmetric to case 2 but since we are constructing left general spanning sets it needs revision. We again construct function $f^{n-1}_{[a',b']}$ where $a' = a^{[1]}a^{[3,n]}$ and $b' = b^{[1]}b^{[3,n]}$. By induction hypothesis there is a left orthogonal set $\mathscr{E}_r$ with desired properties. We construct following set:

$$\mathscr{E} = \{e^{[1]}1e^{[2,n-1]} | e \in \mathscr{E}_r\} \cup \{10(b+1)^{[3,n]}\} \tag{4.3}$$

There is a one to one correspondence between numbers in interval $[a, 2^{n-1}-1]$ and $[a', 2^{n-2}-1]$ and between numbers in interval $[2^{n-1}+2^{n-2}, b]$ and $[2^{n-2}, b']$. By the same argument as in case 2 set $\mathscr{E} \setminus \{10(b+1)^{[3,n]}\}$ is also a left orthogonal set for $f^n_{[a,b]}$. Now let T be a ternary vector spanning both $10(b+1)^{[3,n]}$ and $e \in \mathscr{E} \setminus \{10(b+1)^{[3,n]}\}$. We know that $e^{[2]} = 1$ which implies that $T^{[2]} = \phi$. So T also spans falsepont $b+1$. We have shown that $\mathscr{E}$ is a left orthogonal set for $f^n_{[a,b]}$. Recall that spanning set $\mathscr{T}$ constructed in this case has the size of minimum spanning set of $f^{n-1}_{[a',b']}$ plus one. This fact together with induction hypothesis and the way we have constructed $\mathscr{E}$ implies that $|\mathscr{E}| = |\mathscr{T}|$. By the same argument as in case 2 a right-sided point of $\mathscr{E}_r$ gives us directly a right-sided point of $\mathscr{E}$.

Case 4: We have $a^{[1,2]} = 00$ and $b^{[1,2]} = 11$. Let $j \geq 2$ be the maximum number such that $a^{[1,j]} = 0^{\{j\}}$ and $b^{[1,j]} = 1^{\{j\}}$. Let $a' = a^{[j,n]}$ and $b' = b^{[j,n]}$.

Case 4.1: $b^{[j+1,n]} < a^{[j+1,n]} - 1$. Consider the function $f^{n-j+1}_{[a',b']}$. By induction hypothesis there is a left orthogonal set for this function with desired properties. Denote it by $\mathscr{E}_r$. Now construct a set $\mathscr{S}$ as a set of all j boolean vectors of length n which we get by concatenation of j cyclic shifts of a vector $10^{\{j-1\}}$ (as defined in 2.1) with vector $(b+1)^{[j+1,n]}$. Using this set we define

$$\mathscr{E} = \{c^{\{j\}} e^{[2,n-j+1]} | e \in \mathscr{E}_r \wedge c = e^{[1]}\} \cup \mathscr{S} \tag{4.4}$$

There is a one to one correspondence between numbers in interval $[a, 2^{n-j}-1]$ and $[a', 2^{n-j}-1]$. It is enough to remove first $j-1$ bits to convert a number from the former interval into the latter interval. The same holds for intervals $[2^n - 2^{n-j}, b]$ and $[2^{n-j}, b']$. By the same argument as in case 2 set $\mathscr{E} \setminus \mathscr{S}$ is also a left orthogonal set for $f^n_{[a,b]}$. Now let T be a ternary vector spanning $e_1, e_2 \in \mathscr{S}$ s$(e_1 \neq e_2)$. Both numbers have exactly one 1-bit among their j MSBs. Since $e_1 \neq e_2$ they are on different positions. Therefore T has to span vector $0^{\{j\}}(b+1)^{[j+1,n]}$. Since we assume that $b^{[j+1,n]} < a^{[j+1,n]} - 1$ we know that $0^{\{j\}}(b+1)^{[j+1,n]} < a$ and so it is a falsepoint. Now let T be a vector spanning $e_1 \in \mathscr{S}$ and $e_2 = 0^{\{j\}} e^{[2,n-j+1]}$ for $e \in \mathscr{E}_r(e^{[1]} = 0)$. Such a T has to span a number $0^{\{j\}}(b+1)^{[j+1,n]}$ which is again a falsepoint. At last let T be a vector spanning $e_1 \in \mathscr{S}$ and $e_2 = 1^{\{j\}} e^{[2,n-j+1]}$ for $e \in \mathscr{E}_r(e^{[1]} = 1)$. Such a T has to span a number $1^{\{j\}}(b+1)^{[j+1,n]}$ which is a falsepoint. We have shown that $\mathscr{E}$ is a left orthogonal set for $f^n_{[a,b]}$. Recall that $\mathscr{T}$ was in this case constructed to have size of j plus size of minimum spanning set of $f^{n-j+1}_{[a',b']}$. By our induction hypothesis $\mathscr{E}_r$ has the size of minimum spanning set of $f^{n-j+1}_{[a',b']}$. And thus we have $|\mathscr{E}| = |\mathscr{E}_r| + j = |\mathscr{T}|$. Moreover truepoint $10^{\{j-1\}}(b+1)^{[j+1,n]}$ is a right-sided point of $\mathscr{E}$ since it is its member and $00^{\{j-1\}}(b+1)^{[j+1,n]}$ is a falsepoint.

Case 4.2: $b^{[j+1,n]} \geq a^{[j+1,n]} - 1$. Again consider the function $f^{n-j+1}_{[a',b']}$. By induction hypothesis there is a left orthogonal set $\mathscr{E}_r$ for this function with desired properties. Let $e_r \in \mathscr{E}_r$ be a right-sided point. Construct a set $\mathscr{S}$ as a set of all j boolean vectors of length n which we get by concatenation of j cyclic shifts of a vector $10^{\{j-1\}}$ (as defined in 2.1) with vector $e_r^{[2,n-j+1]}$. We define the set we claim is a left orthogonal set for $f^n_{[a,b]}$ as follows:

$$\mathscr{E} = \{c^{\{j\}} e^{[2,n-j+1]} | e \in \mathscr{E}_r \wedge c = e^{[1]} \wedge e \neq e_r\} \cup \mathscr{S} \tag{4.5}$$

As in case 4.1 there is the same direct one to one correspondence between intervals $[a, 2^{n-j}-1]$ and $[a', 2^{n-j}-1]$ and between intervals $[2^n - 2^{n-j}, b]$ and

$[2^{n-j}, b']$. By the same argument as in case 2 set $\mathscr{E} \setminus \mathscr{S}$ is also a left orthogonal set for $f^n_{[a,b]}$. Now let T be ternary vector spanning $e_1, e_2 \in \mathscr{S}(e_1 \neq e_2)$. By the construction of $\mathscr{S}$ we know that T also span a number $0^{\{j\}}e_r^{[2,n-j+1]}$. Because of e_r being right-sided point of $\mathscr{E}_r$ it is that $0e_r^{[2,n-j+1]}$ is a falsepoint of $f^{n-j+1}_{[a',b']}$ and so $0^{\{j\}}e_r^{[2,n-j+1]}$ is a falsepoint of $f^n_{[a,b]}$ spanned by T. Now let T span numbers $e_1 \in \mathscr{S}$ and $e_2 \in \mathscr{E} \setminus \mathscr{S}$ such that $e_2^{[1,j]} = 0^{\{j\}}$. By construction of $\mathscr{S}$ we have that $e_1^{[j+1,n]} = e_r^{[2,n-j+1]}$. Since e_r is a right-sided point of $\mathscr{E}_r$ we know that $0^{\{j\}}e_r^{[2,n-j+1]}$ is a falsepoint spanned by T. At last let T span numbers $e_1 \in \mathscr{S}$ and $e_2 \in \mathscr{E} \setminus \mathscr{S}$ such that $e_2^{[1,j]} = 1^{\{j\}}$. From construction of $\mathscr{S}$ we see that $e_1^{[j+1,n]} = e_r^{[2,n-j+1]}$. So T necessarily spans a truepoint $1^{\{j\}}e_r^{[2,n-j+1]}$. This corresponds to a truepoint e_r of $f^{n-j+1}_{[a',b']}$. This is the truepoint we exluded from $\mathscr{E}$. So we know that T spans numbers $1^{\{j\}}e_r^{[2,n-j+1]}$ and e_2. That means that ternary vector $1T^{[j+1,n]}$ spans numbers e_r and $e_2^{[j,n]}$. But from the construction of e_2 we know that both of them are members of $\mathscr{E}_r$. Thus there is some falsepoint t of $f^{n-j+1}_{[a',b']}$ of length $n-j+1$ spanned by $1T^{[j+1,n]}$. Then there is a falsepoint $1^{\{j\}}t^{[2,n-j+1]}$ spanned by T. Since $\mathscr{E}_r$ is a left orthogonal set it means that t is falsepoint of $f^{n-j+1}_{[a',b']}$ which is guaranteed not to be spanned by left general spanning set of $f^{n-j+1}_{[a',b']}$. Therefore $1^{\{j\}}t^{[2,n-j+1]}$ is a falsepoint which cannot by spanned by left spanning set of $f^n_{[a,b]}$. We have shown that $\mathscr{E}$ is left orthogonal set for $f^n_{[a,b]}$.

Now recall that in this case $\mathscr{T}$ is constructed to be of size which is $j-1$ plus size of a minimum spanning set of $f^{n-j+1}_{[a',b']}$. By induction hypothesis we know that size of $\mathscr{E}_r$ is the same as size of minimum spanning set of $f^{n-j+1}_{[a',b']}$. So we see that $|\mathscr{E}| = |\mathscr{E}_r| - 1 + j = |\mathscr{T}|$. It remains to show that there is a right-sided point of $\mathscr{E}$. Number $10^{\{j-1\}}e_r^{[2,n-j+1]}$ is a member of $\mathscr{E}$ and $00^{\{j-1\}}e_r^{[2,n-j+1]}$ is a falsepoint of $f^n_{[a,b]}$ because of $0e_r^{[2,n-j+1]}$ being a falsepoint of $f^{n-j+1}_{[a',b']}$.

Note that all falsepoints we have used during proof are quaranteed not to be spanned by ternary vectors in any left general spanning set of $f^n_{[a,b]}$. That completes our proof. $\square$

The difference between this proof and original proof of a theorem 2.1 is that we have used a restricted set of falsepoints of a function. Thus this revised version of a proof is also a valid proof of original version of the theorem with stronger assumptions. Even though the structure of our proof is simpler since we solve the case 4.2 by single argument while original proof needed to split this case into another four cases.

Theorem 4.2. *Let $f^n_{[a,b],[c,d]}$ be a 2-interval function from class B or from class C and let $\mathscr{T}_{opt}$ be its spanning set of minimum size. Moreover let $\mathscr{T}_{approx}$ be a spanning set of f^n returned by approximation algorithm in figure 4.5. Then it holds that*

$$\frac{|\mathscr{T}_{approx}|}{|\mathscr{T}_{opt}|} \leq 2 \tag{4.6}$$

and this bound for an error is tight.

Proof. A lower bound for an error is given by lemma 4.1. For upper bound let $\mathscr{T}$ be a minimum spanning set of $f^n_{[a,b],[c,d]}$. Now let $\mathscr{T}_1$ be spanning set of mininum size of function $f^n_{[a,b]}$ and $\mathscr{T}_2$ be spanning set of mininum size of function $f^n_{[c,d]}$. Without loss of generality let $|\mathscr{T}_1| \geq |\mathscr{T}_2|$. If $f^n_{[a,b]}$ is a prefix function and thus belongs to class B then orthogonal set $\mathscr{E}_1$ of $f^n_{[a,b]}$ produced by proof of lemma 2.1 is generally orthogonal by lemma 4.2. All falsepoints guaranteed not to be spanned by general spanning set are falsepoints of $f^n_{[a,b],[c,d]}$ as well. Thus $\mathscr{E}_1$ is also an orthogonal set of $f^n_{[a,b],[c,d]}$. From proof of lemma 2.1 we know that $|\mathscr{E}_1| = |\mathscr{T}_1|$. By lemma 1.1 size of $\mathscr{T}_1$ is a lower bound on size of minimum spanning set of $f^n_{[a,b],[c,d]}$. Since intervals $[a, b]$ and $[c, d]$ are disjoint also sets $\mathscr{T}_1$ and $\mathscr{T}_2$ are disjoint. So we know that $|\mathscr{T}_1 \cup \mathscr{T}_2| \leq 2|\mathscr{T}|$.

If $f^n_{[a,b]}$ is not a prefix function it belongs to class C. Then $\mathscr{T}$ is also a left general spanning set of function $f^n_{[a,b]}$. By theorem 4.1 $\mathscr{T}_1$ is a left general spanning set of function $f^n_{[a,b]}$ of minimum size. Thus we have $|\mathscr{T}_2| \leq |\mathscr{T}_1| \leq |\mathscr{T}|$. As we have mentioned above sets $\mathscr{T}_1$ and $\mathscr{T}_2$ are disjoint. We conclude that

$$|\mathscr{T}_1 \cup \mathscr{T}_2| \leq 2|\mathscr{T}| \tag{4.7}$$

$\square$

We can enclose this chapter by presenting an algorithm on figure 4.7. It produces a spanning set for 2-interval function. This set is of minimum size if input function belongs to class A. If input function is from class B or C then size of returned set is at most double of minimum size.

approx-finalDNF(a,b,c,d,n)

Input: Numbers a, b, c, d and n such that $0 \leq a \leq b < c \leq d < 2^n$
Output: Spanning set of function $f^n_{[a,b],[c,d]}$

{* *If two intervals are joined to one* *}
if $b + 1 = c$ **then return** *computeDNF-1int*(a,d,n)
{* *If function if of type A* *}
if $a = 0 \wedge d = 2^n - 1$ **then return** *computeDNF-typeA*(b,c,n)
{* *Approximation of functions of type B and C* *}
$\mathscr{T}_1$:= *computeDNF*(a, b, n)
$\mathscr{T}_2$:= *computeDNF*(c, d, n)
return $\mathscr{T}_1 \cup \mathscr{T}_2$

Figure 4.7: Approximation of minimum spanning set for 2-interval function with exact solution for 1-interval function and for 2-interval function of type A

5. Software

We presented some counterexamples or solutions in this thesis which were found with help of a computer. In this chapter we will describe the algorithms we used during research. Their implementation in java is included as attachment in this thesis.

5.1 Maximum orthogonal set finder

This routine is used to find all orthogonal sets of maximum size of given boolean function. Procedure itself works for general functions. However the command line interface allows to enter only 2-interval functions which was enough for our needs. What it does is backtracking over all subsets of truepoints. The number of those are exponential in the number of truepoints. Going through all of them would be wasteful. Luckily it is easy to implement an effective forward checking which eliminates substantial number of branches.

Lemma 5.1. *Let f^n be a function with at least two truepoints. Denote by t_1, t_2 two of its truepoints. Moreover define the following ternary vector of length n:*

$$V^{[i]} = \begin{cases} t_1^{[i]} & \text{if } t_1^{[i]} = t_2^{[i]} \\ \phi & \text{if } t_1^{[i]} \neq t_2^{[i]} \end{cases} \tag{5.1}$$

Then truepoints t_1 and t_2 are orthogonal given f^n if and only if vector V spans some falsepoint of f^n.

forwardChecker(T,F,t)

Input: Set of truepoints T to be filtered, set of falsepoints F, t truepoint to which remaining truepoints should be orthogonal

Output: Set of truepoints such that all of them are orthogonal with t given function defined by falsepoints F

$T' = \emptyset$
for each $tp \in T$ **do**
  {* *Construct vector by lemma 5.1* *}
  $V = getSpanVector(t, tp)$
  if $\exists f \in F : V$ spans f **then** $T' := T' \cup \{tp\}$
enddo
return T'

Figure 5.1: Forward checking routine

Proof. Let t_1 and t_2 be orthogonal given f^n. That means that every vector spannig both of them spans some false point as well. V obviously spans both of them thus it spans some falsepoint.

On the other hand suppose that V spans some falsepoint. Each vector spanning t_1 and t_2 has to have ϕ on each bit where t_1 and t_2 differs and has to have either $t_1^{[i]}$ or ϕ on each bit where they are the same. So it can be seen that each vector spanning both t_1 and t_2 spans a set of numbers which is a superset of numbers spanned by vector V. Thus if V spans some falsepoint then every vector spanning t_1 and t_2 spans some falsepoint as well. $\square$

We can use this simple lemma to construct algorithm which goes through a set of truepoints and removes those which are not orthogonal with given truepoint. We can see this on figure 5.1. This is used as forward checking in backtracking through all possible orthogonal sets. Result can be seen on figure 5.2. We choose a truepoint to include/exclude from orthogonal set under construction on line 4. Recursive call on next line finds all sets without t. t is included into constructed set and all non orthogonal truepoints are removed on lines 6 and 7. Call on line 8 finds all sets with t included. This procedure outputs all possible orthogonal sets (including empty set). It is trivial to choose the one with maximum cardinality.

findOrtSet(T,F,O)

Input: Set of truepoints T such that each of them are orthogonal with O, set of falsepoints F, orthogonal set constructed so far O

Output: Set of all orthogonal sets of function represented by T and F

1: **if** $T = \emptyset$ **then**
2:    report new ort. set O
3: **else**
4:    Let $t \in T$
5:    *findOrtSet*($T \setminus \{t\}$, F, O)
6:    $O' := O \cup \{t\}$
7:    $T' :=$ *forwardChecker*(T, F, t)
8:    *findOrtSet*(T', F, O')

Figure 5.2: Recursive procedure to find all orthogonal sets of given function

5.2 Minimum spanning set finder

In this thesis we have tried to find optimization algorithm which finds a minimum spanning set of an interval function without any searching technique. However, it is usefull to have some procedure which finds the solution using brute force during research. We used backtracking again. The number of all possible orthogonal sets is big but the number of all spanning sets is even bigger. When backtracking through all orthogonal sets we are branching on each truepoint of the function and the decision is whether to include it in orthogonal set or not. Thus branching factor is 2. When backtracking through all spanning sets we are branching on each truepoint as well. But the decision is which ternary vector will be used to span this truepoint. Number of possibilities is exponential in number of variables of a function to be spanned. Therefore if we want to construct reasonably fast algorithm on this basis we need to significantly reduce this exponential branching factor.

Definition 5.1 (Maximal ternary vector)**.** *Let f^n be function and let $T \in \mathscr{T}$ be ternary vector where $\mathscr{T}$ is some spanning set of f^n. We say that T is a maximal ternary vector if it is not possible to change some of its fixed positions ($T^{[i]} \in \{0,1\}$) to ϕ without spanning some falsepoint. We omit word "ternary" if it cannot cause any confusion.*

Observation 5.1. *Each ternary vector can be modified in such a way that it will become a maximal ternary vector for function we are spanning. There are more possible resulting vectors depending on the order in which we try to substitute the positions in it.*

Lemma 5.2. *Each spanning set of a function can be modified in such a way that it contains only maximal ternary vectors.*

Proof. it is enough to take each vector and iterate fixing its positions into ϕ's until the vector is maximal ternary vector. Resulting set is still a spanning set of the same function. □

Using only maximal vectors to span truepoints of a function reduces branching factor. In fact we need to consider all of them if we want to find an optimal solution. That is because for each pair of maximal ternary vectors T_1, T_2 spanning a truepoint of function f^n it holds that

$$(dom(T_1) \setminus dom(T_2) \neq \emptyset) \wedge (dom(T_2) \setminus dom(T_1) \neq \emptyset)$$

That means that each maximal ternary vector is unique in the way how it spans the truepoint and thus cannot be excluded in general from search. We can consider $\phi^{\{n\}}$ to be something like "ultimate" maximal ternary vector. But $\phi^{\{n\}}$ certainly spans some falsepoint (excluding $f(x) = 1$). Falsepoints are the only limitation which determines how maximal vectors look like. Now we are going to furnish procedure producing all maximal ternary vectors for given truepoint of a function.

Lemma 5.3. *Let T_1 and T_2 be a ternary vectors of length n and let I be the set of all indeces i such that $T_1^{[i]} \neq \phi \wedge T_2^{[i]} \neq \phi$. Then it holds that $dom(T_1) \cap dom(T_2) = \emptyset$ if and only if there is some $i \in I$ such that $T_1^{[i]} \neq T_2^{[i]}$.*

Proof. Suppose that there is some $i \in I$ such that $T_1^{[i]} \neq T_2^{[i]}$. This directly implies that there cannot be the number spanned both by T_1 and by T_2. On the other hand suppose that for all $i \in I$ it holds that $T_1^{[i]} = T_2^{[i]}$. Because of this assumption we can correctly define boolean vector V of length n as follows:

$$V^{[i]} = \begin{cases} 0 & \text{if } T_1^{[i]} = T_2^{[i]} = \phi \\ T_1^{[i]} & \text{if } T_1^{[i]} \neq \phi \\ T_2^{[i]} & \text{if } T_2^{[i]} \neq \phi \end{cases} \tag{5.2}$$

It can be easily seen that number V is spanned by both vectors T_1 and T_2. So we see that their domains are not disjoint. □

So we have a function $f^n_{[a,b],[c,d]}$ defined by set of its falsepoints. Moreover we have a truepoint and we want to produce all maximal vectors spanning it. We achieve this using lemma 5.3. Function $f^n_{[a,b],[c,d]}$ has falsepoint intervals $[0, a-1]$, $[b+1, c-1]$ and $[d+1, 2^n-1]$. Some of them may be empty. We can used algorithm 2.4 to produce three spanning sets: $\mathscr{F}_1$ for $f^n_{[0,a-1]}$, $\mathscr{F}_2$ for $f^n_{[b+1,c-1]}$ and $\mathscr{F}_3$ for $f^n_{[d+1,2^n-1]}$. The use of algorithm in figure 2.4 together with lemma 5.3 is a crucial step to reduce a number of spanning sets we shall go through during backtracking. One of properties of spanning set we need to maintain is that it does not span any falsepoint. Now for spanning set $\mathscr{T}$ we can formulate this property as

$$dom(\mathscr{T}) \cap dom(\mathscr{F}) = \emptyset \tag{5.3}$$

where $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$. To find maximum ternary vector spanning truepoint t for function $f^n_{[a,b],[c,d]}$ it means to find all maximum vectors fulfiling the condition

$$dom(t) \cap dom(t') = \emptyset \text{ for all } t' \in \mathscr{F} \tag{5.4}$$

We can represent all necessary information about falsepoints of a function and a truepoint t to be spanned as a graph $\mathscr{G}$ defined in a following way:

$$V(\mathscr{G}) = \{V | V \in \mathscr{F}\} \cup \{"x^{[i]} = c" | 1 \leq i \leq n \wedge c \in \{0,1\}\} \tag{5.5}$$

$$E(\mathscr{G}) = \{\{V, "x^{[i]} = c"\} | t_i^{[i]} = c \wedge x_i^{[i]} = c \Rightarrow t \notin dom(V)\} \tag{5.6}$$

It is a bipartite graph where one part of verteses represents vectors from set $\mathscr{F}$. Second part represents all possible predicates about value of each of n bites of ternary vector being constructed. Edge is only between "the vector" and "the predicate" and only if the predicate implies that "the vector" does not span truepoint t. So all edges incident to vertex of a ternary vector $V \in \mathscr{F}$ represent the list of all possibilities how ternary vector under construction can avoid to span something from $dom(V)$. Usage of this graph for finding all maximal vectors spanning given truepoint can be seen in figure 5.3. Procedure presented in it simply goes through all combinations of vectors. It tries to fix suitable positions in ternary vector under construction in such a way that domain of resulting vector is disjoint to domain of every vector spanning some falsepoint interval. With this as a subroutine it is not hard to finally implement algorithm generating all spanning sets of a function consisting from maximal vectors. It is displayed on figure 5.4. It is easy to choose the one with mimimum cardinality.

findMaxVectors($\mathscr{G}$,$\mathscr{V}$,$outV$)

Input: Graph $\mathscr{G}$, vectors from spanning set of falsepoints $\mathscr{V}$ to be avoided, maximal ternary vector $outV$ under construction

Output: Set of all maximal ternary vectors induced by graph $\mathscr{G}$ and vectors in $\mathscr{V}$

if $\mathscr{V} = \emptyset$ **then** report new max. vector $outV$ and **return**
$V \in \mathscr{V}$
{** If we already solved this vector by some preceding assigement we can move on **}
if $\exists "x^{[i]} = c" : outV^{[i]} = c \wedge \{V, "x^{[i]} = c"\} \in V(\mathscr{G})$ **then**
  lookForVectors($\mathscr{G}$,$\mathscr{V} \setminus \{V\}$,$outV$)
  return
{** If it is not solved we try all satisfying assigments **}
for each $"x^{[i]} = c" : \{"x^{[i]} = c", V\} \in E(\mathscr{G})$ **do**
  {** If it is not already fixed to different value **}
  if $outV^{[i]} = \phi$ **then**
    $outV' := outV^{[1,i-1]} c\, outV^{[i+1,n]}$
    lookForVectors($\mathscr{G}$,$\mathscr{V} \setminus \{V\}$,$outV'$)
enddo

Figure 5.3: Algorithm for generation of all maximal ternary vectors spanning truepoint of a function

findSpanSet($\mathscr{S}$,$\mathscr{T}$,$\mathscr{F}$,max)

Input: Spanning set $\mathscr{S}$ constructed so far, Set of truepoints $\mathscr{T}$ to be spanned, Spanning set $\mathscr{F}$ of falsepoints of f^n, maximum allowed number of vectors in $\mathscr{S}$

Output: Set of all spanning sets of f^n consisting from maximal vectors

if $\mathscr{T} = \emptyset$ **then** report new spanning set $\mathscr{S}$ and **return**
if $max = 0$ **then return**
$t \in \mathscr{T}$
{** Construct graph as in 5.5 and 5.6 **}
$\mathscr{G}$:=*generateGraph*($\mathscr{F}$, t)
spanCand:=*findMaxVectors*($\mathscr{G}$, $V(\mathscr{G})$, $\phi^{\{n\}}$)
for each $V \in$ *spanCand* **do**
  $\mathscr{T}' := \{T | T \in \mathscr{T} \wedge T \notin dom(V)\}$
  findSpanSet($\mathscr{S} \cup \{V\}$, $\mathscr{T}'$, $max - 1$)
enddo

Figure 5.4: Algoritm for generation of minimum spanning set of function

6. Conclusion

In this thesis we have studied a construction of DNF representations of 1-interval and 2-interval functions. We presented known results for class of 1-interval functions. All of them were proved using method of comparing sizes of some spanning set and some orthogonal set. We formalised this method and studied its possibilities. Using it we have constructed an optimization algorithm for 2-interval functions of type A. Counterexamples were found proving weakness of this method for general 2-interval functions. Thus we have turned our attention to approximation algorithms. Alternative way of proving the main theorem from [2] were presented. Approximate algorithm were constructed and its exact error analysed.

Furthermore we have provided in figures few examples of functions which we believe are important for next study of the field. Simple software for finding spanning sets of minimum size and orthogonal sets of maximum size was developed. It is based on search methods but it is optimised so it can be used with suitable hardware for experimenting with functions of up to seven variables. We believe we have contributed into knowledge about minimum DNF representation construction for interval functions and that our work will be of good basis for another research in this field.

List of Figures

Bibliography

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